Scattering in Asymptotically Flat Spacetimes and Symmetries from Holography

A thesis submitted in partial fulfilment of the requirements

for the award of the degree of

DOCTOR OF PHILOSOPHY

by

TABASUM RAHNUMA

1910512

to the

DEPARTMENT OF PHYSICS

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL

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CERTIFICATE

The undersigned have examined the Ph.D. thesis entitled:

Scattering in Asymptotically Flat Spacetimes
and Symmetries from Holography

presented by Ms.Tabasum Rahnuma, a candidate for the degree of Doctor of Philosophy in the department of Physics, and hereby certify that it is worthy of acceptance.

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"It wasn't the New World that mattered... Columbus died almost without seeing it; and not really knowing what he had discovered. It's life that matters, nothing but life — the process of discovering, the everlasting and perpetual process, not the discovery itself, at all."

- Fyodor Dostoevsky, The Idiot

DEDICATION

This thesis is dedicated to my family

(Ammi, Abbu, Jahed, and my dearest Nani)

and

to all those who fight their battles with strength, hope, and love in their hearts.

ABSTRACT

The gravitational dynamics in flat spacetime can be understood through the holographic principle, a concept proposed by Gerard 't Hooft and later by Leonard Susskind and Juan Maldacena, who played a pivotal role in the development of the principle. In the 1960s, Bondi, van der Burg, Metzner, and Sachs (BMS) delved into understanding the intricate symmetries of flat spacetimes, revealing deeper insights into the spacetime structures. Most recently, Andrew Strominger highlighted the principle and used these symmetry understanding to conjecture *Celestial Holography*, establishing a holographic correspondence between the quantum gravity in $4D$ asymptotically flat spacetimes and a $2D$ conformal field theory at the boundary.

In this thesis, we intend to analyze the scattering in Asymptotically flat spacetimes and the symmetries at the asymptotics with a new understanding of scattering in the flat space limit of Anti-de-Sitter (AdS) spacetimes.

Symmetries From (*Celestial*) Holography

Our focus here is on the holographic technique for Asymptotically Flat Spacetimes (AFS), which maps the fields from the bulk of the spacetime to the boundary conformal operators sitting at the *celestial* sphere of the null boundaries of our asymptotically flat spacetimes.

Recent developments in Celestial Holography have suggested the relevant collinear celestial amplitudes to ascertain asymptotic symmetries in $4D$ AFS. We used the techniques of celestial holography to find the asymptotic symmetry algebras for 4D Einstein-Yang-Mills and Einstein-Maxwell theories recovering the complete local superrotation algebra.

In the other line of work, we found the asymptotic symmetries of $\mathcal{N} = 8$ Supergravity. The global symmetry algebra of the theory consists of $\mathcal{N} = 8$ superpoincarè algebra and $\mathrm{SU}(8)_R$ symmetry algebra. Our study indicates that the asymptotic *soft hairs* of $\mathcal{N} = 8$ supergravity theory will not have distinct infinite R-charges. Instead, they will only carry the global fixed number of R-charges.

Double Copy in Universal Sectors of Scattering

The Double Copy (DC) is a multiplicative bilinear operation to compute the amplitudes of gravity tree-level amplitudes in terms of sums of products of gauge theory tree-level amplitudes. This well-known technique has wide applications in quantum and classical field theories, including string theory, particle physics, and astrophysics. In our analysis, we restrict the DC formalism to soft and collinear sectors of amplitudes in both gravity and gauge theory.

We established the non-trivial relationships between the amplitudes of these two theories using DC formalism in *celestial* basis. Soft and collinear sectors of $\mathcal{N} = 4$ Super Yang-Mills (SYM) can be double copied individually to result in the soft and collinear sectors of $\mathcal{N} = 8$ Supergravity. Our goal in this work was to construct the dual *celestial* CFT (CCFT) corresponding to the bulk $\mathcal{N} = 8$ supergravity in four spacetime dimensions.

Scattering in the flat limit of AdS

$\mathcal{S}$-matrix is a well-defined observable for quantum field theories in flat space-time. However, for theories in AdS space-time, the $\mathcal{S}$-matrix is not well-defined. In AdS, particles correspond to irreducible representations of the conformal group, allowing for a connection between QFT in flat space and QFT in AdS. This connection is established through the relationship that the $\mathcal{S}$-matrices of flat space can be derived by taking the dimension of the conformal field (Δ) in the CFT correlator to be large when the dual AdS length scale (AdS radius $\to \infty$ limit). This transition enables a comprehensive understanding of QFT in AdS by utilizing insights from CFT correlation functions and their connection to the flat space formulation. Here, we addressed the question of finding the '*Scattering Matrix*' for the AdS in this flat space limit.

Our primary goal is to understand the Infrared (IR) behavior of this defined $\mathcal{S}$-matrix in the flat space limit of AdS/CFT. We follow the Momentum space prescription, to define the 'AdS $\mathcal{S}$-Matrix' as the Fourier transform of the position space correlation function in the embedding space. This encodes all the information on all of the bulk physics in the conformal correlator in $1/R$ perturbation theory. We concluded this work with the computations of bulk-to-bulk and bulk-to-boundary propagators of vector particles.

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CHAPTER 1

INTRODUCTION

Solving a theory implies computing the physical observables encoded in the symmetries of the theory, and these observables are the scattering amplitudes. On the other hand, the nature of the scattering amplitudes in Quantum Field Theory (QFT) is fundamental in discovering non-trivial symmetries of the theory. Every possible symmetry in the theory provides important constraints on the form and behavior of scattering amplitudes and other observables. This is ensured by the Ward identities, which are manifestations of symmetries in QFT.

Furthermore, it has been observed that in gauge and gravity theories, there is an enhancement of symmetries at the boundaries of the spacetimes. For such asymptotic boundaries, these enhanced symmetries are known as the *asymptotic symmetries* of the spacetime. In four dimensions, these asymptotic symmetries have been studied for both gauge and gravity theories, including supergravity [1–7]. These symmetries in the case of pure gravity and gauge theories are popularly known as BMS (Bondi–van der Burg–Metzner–Sachs) and *large* gauge symmetries, respectively with some enhancements depending on the kind of theory in considerations [8–16]. These infinite dimensional asymptotic symmetries also have experimental implications as in gauge and gravitational memory effects which are the classical observables [17–21]. In the subsections below, I will highlight the features of Asymptotically Flat Spacetimes (AFSs), along with brief studies of scattering processes and symmetry constraints.

Considering the motivation of these symmetry studies for gauge and gravity theories, the soft theorems, infinite-dimensional asymptotic symmetries, and the memory effects are the three different manifestations of a single framework. This is wonderfully portrayed in Strominger's Infrared (IR) triangle [11]. I will explain the state of the art and applications of this incredible observation later in the subsections.

Below, I will start with the definition of Asymptotically Flat Spacetimes and motivate the reader to study these specific spacetimes in four dimensions. I will then dive into the detailed descriptions of symmetry along with the algebra constructions. In the end, I will explain how one can use holography to describe the properties of the spacetimes, which, in our case, contains new and exciting applications.

1.1 Understanding Physics from Asymptotics

In theoretical physics, asymptotics of the spacetimes refers to the behavior of spacetime at great distances from the sources. These sources can be gravitational or electromagnetic fields. These studies are important considering the global structure and properties of our universe, especially the properties of black holes.

Below, I will explain some of the aspects related to the asymptotically flat spacetimes in $(3+1)$ spacetime dimensions. In $d \leq 3$ dimensions, there are no local gravitational degrees of freedom (Gravity is *topological*!). Amongst the most studied solutions in $3D$ gravity with a negative cosmological constant is the BTZ (Banados-Teitelboim-Zanelli) black hole [22–24]. In the $(1+1)$dimension, gravity is topological as well, with no dynamic solutions to Einstein's equation of motion. We need to consider the dilaton gravity (*JT Gravity*[1]) to include the dynamics.

In this work, we will focus exclusively on $4D$ flat spacetimes.

1.1.1 $4D$ Asymptotically Flat Spacetimes (AFS)

Asymptotically flat spacetimes (AFS) in $3+1$ dimensions are rich in physics for describing the events in gravity in astrophysical time scales. These are the solutions to the Einstein field equations, which are good approximations of our observable universe[2]. We study the $4D$ quantum gravity scattering $\mathcal{S}$-matrix and the symmetry constraints in our desired class of asymptotically flat spacetimes. Since the sixties, there have been seminal works to study symmetries of this kind of spacetimes [4, 15]. These are the solutions to Einstein's equations for vanishing cosmological constants $(\Lambda = 0)$. One can introduce Penrose compactification to best describe the asymptotic infinities of Minkowski spacetime [31–34], and this is shown in Fig. 1.1.

This compactification conformally maps our spacetimes to a Lorentzian manifold with finite extent and boundary that is differentiable everywhere [35]. In our prescription, we will consider the full deduction of suitable boundary conditions for AFSs. However, one can also consider the knowledge of AdS spacetimes in flat space limit, which means simply taking $(\text{Radius}(\ell) \to \infty)$ limit, which fulfills our demand of flat spacetimes. I will talk about both the classes of spacetimes in this work. For the time being, let us define our desired class of spacetimes below.

[1] In these theories, a scalar field, which is our dilaton field, couples to gravity to generate dynamics to the gravitational field [25–27]. This $2D$ model offers applications in quantum gravity and holography, while $3D$ models reveals features of conical singular solutions in classical and quantum gravity [28–30].

[2] As the curvature radius is of the order of 10^{60} in the Planck units.

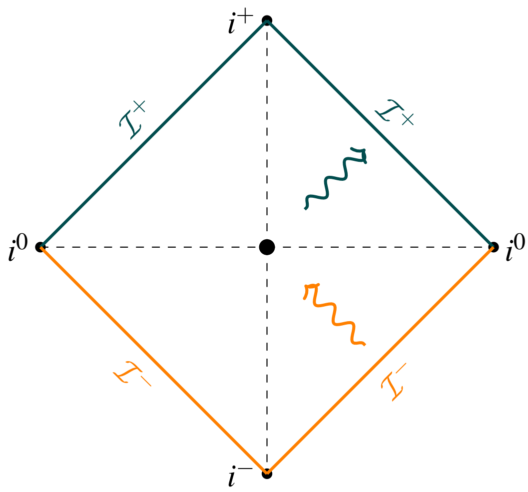


Figure 1.1: Complactified Mink$_4$ Spacetime. The null boundary of the spacetimes has been divided into past (orange) null infinities of $\mathcal{I}^-$ and future (teal) null infinities of $\mathcal{I}^+$. Here, the null (radiations) rays (wavy lines) come all the way from $\mathcal{I}^-$ to $\mathcal{I}^+$. i^0 and $i^\pm$ are the spacelike and timelike boundaries respectively. In the massless scattering, we will study the emitted radiation particles that will reach our null boundaries.

As I said, to study the asymptotic infinities of the Minkowski spacetime, we need Penrose conformal compactification. This compactified spacetime can be written in terms of the spherical coordinates (t, r, x^A), where the boundaries are at $r = \infty$. The metric is given by

$$ds^2 = -dt^2 + dr^2 + r^2\, \gamma_{AB}\, dx^A\, dx^B. \tag{1.1}$$

Here, γ_{AB} is the metric on the two-sphere with coordinates $x^A = (\theta, \phi)$ sitting at each and every point of the compactified spacetime, except at the center ($r = 0$), as shown in Fig.1.1.

Now, we will introduce retarded and advanced coordinates parametrizing the null boundaries of the spacetimes, $u = t - r$ on $\mathcal{I}^+$, and $v = t + r$ on $\mathcal{I}^-$, respectively.

Geometrically, these spacetimes can be explained by the metric $(g_{\mu\nu})$ in the *Bondi-Sachs coordinates*[3] (u, r, x^A), where u is the retarded time, r is the radial coordinates or the affine parameter along the null geodesics at constant u hypersurfaces. The solution reads as,

$$ds^2 = -du^2 - 2du dr + r^2 \gamma_{AB} dx^A\, dx^B, \tag{1.2}$$

where, γ_{AB} is the unit round metric on the 2−sphere. Now, we can define the metric at the

[3]not unique [36].

asymptotic ($r \to \infty$) as

$$ds^2|_{r\to\infty} = \left(-1+\frac{2m_B}{r}\right)du^2+2\left(-1+\mathcal{O}(1/r^2)\right)dudr+\left(r^2\gamma_{AB}+\mathcal{O}(1)\right)dx^A\,dx^B, \\ +rC_{AB}dx^Adx^B+\left(D^BC_{AB}+\frac{1}{r}\left(\frac{4}{3}(N_A+u\partial_A m_B)-\frac{1}{4}\partial_B\left(C_{DE}C^{DE}\right)\right)\right)dudx^A+\dots. \tag{1.3}$$

These spacetimes are obtained when we take the $r \to \infty$ limit of the Minkowski spacetime at constant u and x^A. Here, we demand certain boundary conditions on the metric components at the asymptotics, which can incorporate all the physical spacetimes.

Here, we define

- $m_B(u,x^A)$ is the *Bondi mass aspect*, which defines the total angular energy density of the spacetime measured by an observer at a particular time u along x^A direction. After integrating over the sphere, this results in the total Bondi mass.

- $N_A(u,x^A)$ is called the *Bondi angular momentum aspect*, which measures the total angular momentum density of the spacetime with respect to the origin ($r = 0$) in the Penrose diagram. After integrating over the sphere, we get the total angular momentum.

- $C_{AB}(u,x^A)$ is a traceless field tensor ($\gamma^{AB}C_{AB}$) encoding the gravitational radiation information. This component of the metric is transverse to the null boundary.

- D^A is the covariant derivative with respect to the metric γ^{AB} on the unit sphere S^2.

The metric solutions that satisfy this kind of boundary fall-offs of the metric components at $r \to \infty$ as given in Eq. (1.3) are our desired asymptotically flat metrics. Specifically, they are [35],

$$g_{uu} = -1+\mathcal{O}\left(\frac{1}{r}\right),\ g_{ur} = -1+\mathcal{O}\left(\frac{1}{r^2}\right),\ g_{uA} = \mathcal{O}(1),\ g_{AB} = r^2\gamma_{AB}+\mathcal{O}(r). \tag{1.4}$$

Here, we used a particular gauge condition that fixes all the local diffeomorphisms at the asymptotics, which is famously called *Bondi Gauge* conditions and is given by

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0,\ g_{rr} = g_{rA} = g_{AB} = 0. \tag{1.5}$$

Let us say we have some matter stress tensor $T^M_{\mu\nu}$, and the geometry of the spacetime is governed by Einstien's equations,

$$R_{\mu\nu} - \frac{1}{2} g_\mu R = 8\pi G T^M_{\mu\nu}. \tag{1.6}$$

Now, once we substitute the metric defined in Eq.(1.3) in Einstein's equations, we get some constraint equations [35],

$$\begin{cases} \partial_u m_B = \frac{1}{4} D^A D^B N_{AB} - T_{uu}, \\ \partial_u N_A = -\frac{1}{4} D^B \left(D_B D^E C_{FE} - D_F D^E C_{BE} \right) + u \partial_A \left(T_{uu} - \frac{1}{4} D^B D^E C_{BE} \right) - T_{uA}, \end{cases} \tag{1.7}$$

where

$$\begin{cases} T_{uu} = \frac{1}{8} N_{AB} N^{AB} + 4\pi \lim_{r\to\infty} \left(r^2 T^M_{uu} \right) \\ T_{uA} = 8\pi \lim_{r\to\infty} \left(r^2 T^M_{uA} \right) - \frac{1}{4} \partial_A \left(C_{EF} N^{EF} \right) + \frac{1}{4} D_E \left(C^{EF} N_{FA} \right) - \frac{1}{2} C_{AB} D_E N^{BE} \end{cases}$$

These constraints show that the boundary data are specified by the values of m_B, C_{AB}, N_A, and N_{AB} at $\mathcal{I}^\pm$.

In this thesis, we use the stereographic complex coordinates $(z, \bar{z})$ on the sphere, which is helpful in the study of celestial holography.

Stereographic coordinates:

$$(\theta, \phi) \to (z, \bar{z}); \quad z = \cot\frac{\theta}{2} e^{i\phi}, \ \bar{z} = \cot\frac{\theta}{2} e^{-i\phi},$$

Such that the metric on the sphere becomes,

$$\gamma_{AB} dx^A \, dx^B = 2\gamma_{z\bar{z}} \, dz \, d\bar{z},$$

where[4], $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ is the round metric on the S^2 in the coordinates defined $(z, \bar{z})$. We interpret $z = 0$ and $z = \infty$ as the north and south poles of the sphere, respectively, with the equator defined at $z\bar{z} = 1$. The points at the past and future null infinities are identified using the *antipodal map* defined as $z \to -\frac{1}{\bar{z}}$, which is important in defining the continuity of fields at the spatial infinities

[4] we have chosen the phase factor $\phi = 0$ [37].

[11]. This relates to the boundary data of past and future null infinities. Hence, it is important to consider the equivalence of BMS charges across spatial infinity.

In this coordinate system, our asymptotically flat metric defined in Eq.(1.3) becomes,

$$
\begin{aligned}
ds^2|_{r\to\infty} = & \left(-1+\frac{2m_B}{r}\right)du^2 + 2\left(-1+\mathcal{O}(1/r^2)\right)dudr + 2\left(r^2\gamma_{z\bar{z}} + \mathcal{O}(1)\right) dz\, d\bar{z}, \\
& + \left[rC_{zz}dz^2 + \left(D^zC_{zz} + \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z\left(C_{zz}C^{zz}\right)\right)\right)dudz + \text{c.c.}\right] + \dots .
\end{aligned}
\tag{1.8}
$$

Next, we need to define our asymptotic phase space with a few restrictive boundary conditions that will allow for all of our physical spacetimes. Hence one can find the boundary falloffs at $r \to \infty$ in stereographic coordinates as [11],

$$
\begin{cases}
g_{uu} = -1 + \mathcal{O}\left(\frac{1}{r}\right), \\
g_{ur} = -1 + \mathcal{O}\left(\frac{1}{r^2}\right), \\
g_{uz} = \mathcal{O}(1), \; g_{zz} = \mathcal{O}(r), \\
g_{z\bar{z}} = r^2\gamma_{z\bar{z}} + \mathcal{O}(1), \quad g_{rr} = g_{rz} = 0.
\end{cases}
\tag{1.9}
$$

In the next section, we will see that the symmetry group at the asymptotics is larger than the exact symmetry group of the flat spacetimes in the bulk by studying the scattering in AFSs. Before that, let us briefly discuss the radiation fluxes that one can observe at the boundary of spacetime.

Radiation Fluxes

The *radiation zone* of asymptotically flat spacetimes is the future null infinity $\mathcal{I}^+$, which is topological $\mathbb{S}^2 \times \mathbb{R}$, where we can collect all the information about the null rays and gravitational waves (GWs).

As one can see in Eq.(1.3), we have the information about the gravitational radiations encoded in the field tensor $C_{zz}(u,z,\bar{z})$ [11, 35, 38]. This is a traceless tensor that implies the two physical degrees of freedom $(C_{zz}, C_{\bar{z}\bar{z}})$ corresponding to the GWs. The physical observable is the change of this field along u direction represented by the *Bondi News* tensor [38], $N_{zz} = \partial_u C_{zz}$. This is the analog of the electromagnetic field strength in electrodynamics ($F_{uz} = \partial_u A_z$), and the square, the energy flux across the null infinities relates to the Gravitational energy flux

[11]. The asymptotic metric given in Eq.(1.3) is the solution to Einstein's equation, which constrains m_B and $N_{z(\bar{z})}$ as shown in Eq.(1.7). This implies our boundary radiation data at $\mathcal{I}^{\pm}$ are $\{m_B, C_{zz}, C_{\bar{z}\bar{z}}, N_{z(\bar{z})}\}$.

1.2 Scattering In 4D AFSs

Symmetries in a quantum field theory are the most important features. Solving a theory means computing the scattering amplitudes in terms of the correlation functions[5]. This is often dictated by the number of symmetries the theory possesses, which puts constraints on the correlation functions. These symmetries in the theory are reflected in the scattering amplitudes via the Ward identities. This is called bootstrapping and has been a very effective tool in conformal field theory [40–42]. This also goes in the reverse direction; that is, knowledge about the nature of amplitudes can help us discover non-trivial symmetries of the theory. I will focus on the scattering processes in 4D Mikowski spacetimes (Mink$_4$). So, our motivation is to answer the question,

How asymptotic symmetry constrain dynamics at $\mathcal{I}^{\pm}$ for massless scattering in Mink$_4$?

Symmetries play important roles in theories coupled to gravity in scattering processes for Mink$_4$. In all these works, we are solely focused on studying the dynamics of the null boundaries of compactified (Mink$_4$) for the scattering of massless particles, and I will discuss the gauge-invariant on-shell scattering amplitudes.

1.2.1 Scattering $\mathcal{S}$-Matrix in Gauge theory

Let us demonstrate an example of scattering in Quantum Electrodynamics (QED) (Abelian Gauge theory). In this theory, we have the symmetry as global $U(1)$ along with the global Poincaré symmetry in the absence of any massless charged particles, and corresponding conserved charges can be measured at the spatial and null boundaries of the spacetime. Now, once we introduce some massless charged particles in our system, we will encounter charge fluxes throughout the null boundaries. At each point of $\mathcal{I}^{\pm}$, we have an S^2 sitting which is associated with many different charges (cf. [11, Section 2: QED]). This means the symmetry of $\mathcal{I}^{\pm}$ is not

[5] Generically, this can be done using the LSZ formulation. In celestial holography, in terms of the conformal basis, we use an alternative prescription to LSZ to extract the massless scattering $\mathcal{S}$-matrix. The in/out states are computed by integrating the gauge field or the metric along the null ray at past and future null infinities of our AFS. Conformal operators of different conformal dimensions are prepared with the operators integrated along the null rays, and the correlation function of these operators gives the $\mathcal{S}$-matrix elements for any massless theory [39].

any more global $U(1)$. The non-trivial charge fluxes in the presence of charged particles at $\mathcal{I}^{\pm}$ enhance this global $U(1)$ to infinite dimensional *large* $U(1)$ gauge symmetry transformations at the *sky* of our Mink$_4$. Here, by *sky*[6], I mean the full null boundaries of our conformally compactified Minkowski spacetime as shown in Fig.1.1.

Now the question comes

What is this infinite-dimensional large gauge transformations?

Any gauge theory has gauge redundancies, which are unphysical degrees of freedom of the theory. We must fix these redundant gauge freedoms to make the theory physically relevant[7]. Our gauge field transforms (in the Bondi parameter space) as $A_z(u,z,\bar{z}) \rightarrow A_z(u,z,\bar{z}) + \partial_z \varepsilon(z,\bar{z})$. After imposing the Lorentz gauge condition, we can have some leftover residual gauge transformations[8]. These leftover transformations are our *large* gauge symmetry transformations, which are locally relevant physical solutions of the gauge vector fields, which create the zero-momentum (*soft*) photons at the boundary. So, these are the asymptotic symmetries in a gauge theory. One can find the explicit derivations to asymptotic symmetries and transformation in the work by Strominger et al. here in [6, 11].

This means *large* gauge symmetry transformations are represented by functions $\varepsilon(z,\bar{z})$ parametrized on the conformal sphere at $\mathcal{I}^{\pm}$. In case of scattering in asymptotically flat Mink$_4$, we have a bunch of incoming photons coming all the way from the $\mathcal{I}^{-}$ creating matter charge current J^{μ}_{M} outgoing towards $\mathcal{I}^{+}$. However, one can find discontinuity of the spacetime at the spatial infinity i^0, which implies the discontinuity of the (leading) electromagnetic field strength tensor near i^0. This issue has been taken care of by the Lorentz invariant *anti-podal matching* between the spheres sitting at the past of $\mathcal{I}^{+}(\mathcal{I}^{-}_{+})$ and the future of $\mathcal{I}^{-}(\ \mathcal{I}^{+}_{-})$ near i^0 [11]. Hence, this implies a smooth functional parametrization on the $2D$ conformal sphere, which is also anti-podally identified. This concludes that we have infinite charge conservation for every function $\varepsilon(z,\bar{z})$ on the S^2 at each point $(z,\bar{z})$ of the boundary: $Q^{+}_{\varepsilon} = Q^{-}_{\varepsilon}$.

Quantum scattering amplitude is written as $\langle out|\mathcal{S}|in\rangle$, where $\mathcal{S}$ denotes the scattering $\mathcal{S}$-

[6]In the language of Celestial holography, we frequently use the word 'celestial' referring to the sky or the boundary of our spacetime.
[7]This is analogous to residual diffeomorphism in gravity acting on the metric field $g_{\mu\nu}$.
[8]This corresponds to the solutions of $\Box\varepsilon = 0$ at the boundaries of $\mathcal{I}^{\pm}$.

matrix. Then, we have the statement of charge conservation as follows,

$$\langle \text{out}|Q^{+}_{\varepsilon}\mathcal{S}-\mathcal{S}Q^{-}_{\varepsilon}|\text{in}\rangle = 0. \tag{1.10}$$

In retarded coordinates, we have the charge [11],

$$Q^{+}_{\varepsilon} = \frac{1}{e^2}\int_{\mathcal{I}^{+}_{-}} dz^2\, \gamma_{z\bar{z}}\varepsilon F^{(2)}_{ru} \tag{1.11}$$

Here, $F^{(2)}_{ru}$ is the $1/r^2$ order term in the expansion of ru component of the electromagnetic field strength tensor around the null infinity. After a bit of simplification using the constraint from the $r\to\infty$ of Maxwell's equations ($\nabla^{\mu}F_{\mu\nu} = e^2 j_{\nu}$), we get the simplified expression for charge as [11],

$$Q^{+}_{\varepsilon} = -\frac{1}{e^2}\int_{\mathcal{I}^{+}} dudz^2\left(\partial_z\varepsilon\, F^{(0)}_{u\bar{z}} + \partial_{\bar{z}}\varepsilon\, F^{(0)}_{uz}\right) + \int_{\mathcal{I}^{+}} dudz^2\varepsilon\gamma_{z\bar{z}}j^{(2)}_{u}. \tag{1.12}$$

The above expression implies that, the localized charge at each point $(z,\bar{z})$ of the sphere is factorized as,

$$Q^{\pm}_{\varepsilon} = Q^{\pm(\text{Soft})}_{\varepsilon} + Q^{\pm(\text{Hard})}_{\varepsilon} \tag{1.13}$$

Here, the first term in Eq.(1.12) is *soft* charge, which is linear in the electromagnetic field, and the nonlinearity of the electromagnetic field and the charge current is included in the second term as the *hard* part of the charge.

Suppose in the scattering, we have N_1 number of incoming and N_2 number of outgoing particles of charges Q^{in}, and Q^{out}, respectively. Schematically the action of this Soft/Hard charge at $\mathcal{I}^{\pm}$ for each value of $\varepsilon(z,\bar{z})$ on the in-state are given by [11],

$$\begin{aligned} Q^{-(\text{Soft})}_{\varepsilon}|\text{in}\rangle &= 2\int d^2z\, \partial_z\varepsilon(z,\bar{z})\, \partial_{\bar{z}}N^{-}(z,\bar{z})|\text{in}\rangle, \\ Q^{-(\text{Hard})}_{\varepsilon}|\text{in}\rangle &= \sum_{m=1}^{N_1} \text{Q}^{\text{in}}_{m}\, \varepsilon(z^{\text{in}}_{m},\bar{z}^{\text{in}}_{m})|\text{in}\rangle. \end{aligned} \tag{1.14}$$

similarly, we have the action on the out-state. Here, the soft charges are the integrated value of the soft photon field $N^{\pm}(z,\bar{z})$ on the sphere, defined as $\partial_z N \equiv \frac{1}{e^2}\int_{-\infty}^{+\infty} duF^{(0)}_{uz}$.

From Eq.(1.10), one can see the *infinite* number of Ward identities (corresponding to each value of ε). Hence, we can write,

$$\langle \text{out}|Q_\varepsilon^{+(\text{Soft})}\mathcal{S} - \mathcal{S}Q_\varepsilon^{-(\text{Soft})}|\text{in}\rangle = \Big[\sum_{m=1}^{N_1} \mathrm{Q}_m^{\text{in}}\, \varepsilon(z_m^{\text{in}}, \bar{z}_m^{\text{in}}) - \sum_{n=1}^{N_2} \mathrm{Q}_n^{\text{out}}\, \varepsilon(z_n^{\text{out}}, \bar{z}_n^{\text{out}})\Big] \langle \text{out}|\mathcal{S}|\text{in}\rangle \quad (1.15)$$

This says that the conserved charges corresponding to these symmetries commute with the $\mathcal{S}$-matrix. These statements are precisely the statements of *soft photon theorems* of asymptotic scattering in case of our abelian $U(1)$ gauge theory[9][45, 47–57]. Although soft theorems and asymptotic symmetries are studied independently, in 2010, Strominger discovered the intriguing relations between these two, which are mathematically related by Ward identities of asymptotic symmetries of a theory [12, 15, 47].

In the language of 2D CFT on the sphere, the symmetry current corresponding to this *large* gauge transformations are the $U(1)$ Kac-Moody current on the two-sphere at the null boundary [6, 58–61].

Now, let us talk about an infinite number of *soft* gluon emissions in the scattering process in non-abelian gauge theory.

Non-Abelian Gauge Theory:

Conserved asymptotic gauge-invariant electric and magnetic charges are defined for non-abelian gauge theories for asymptotic field configurations [62–66]. Non-abelian gauge theory (e.g., Pure Yang-Mills Theory[10] is not confined and asymptotically free [11, 67–69]. Hence, the asymptotic detectors can observe the *soft* gluons.

In *perturbative* computation [67, 70], the asymptotic states are formed by the action of the creation and annihilation operators of the hard external particles of momenta p_i,

$$\prod_i a^\dagger_{c_i,h_i}(p_i)|0\rangle = |\Omega_{c_i,h_i}(p_i)\rangle \quad (1.16)$$

where, c_i and h_i are the color index and the helicity of the ith particle.

We can interpret this Fock vacuum as the *dressed*[11] vacuum. Now an N-particle IR diver-

[9] These are the statements for scattering amplitudes associated with zero-momentum (soft) photon emissions [12, 43–46]. These statements are related by Fourier transformation to the Electromagnetic memory effects [20].

[10] In case of hadronic quark-gluon model of QCD with an $\mathrm{SU}(3)_{color}$, quarks are not asymptotically free, which means color symmetry is an exact symmetry, hence because of the color-singlet bound states, hadronic amplitudes should have no infrared singularities [67].)

[11] This Fock vacuum is the eigenstate of the linearized charge with zero eigenvalues. This can be understood in the Faddeev-Kulish (FK) approach to the perturbative theory in the non-abelian case. The hard external parton

gent *perturbative* $\mathcal{S}$-matrix element of the Yang-Mills theory between these asymptotic states is given by,

$$\begin{aligned}\mathcal{M}_N(\{p_i,h_i,c_i\}) &= \langle\{p_i,h_i,c_i\}\big|_{i\in\text{out}}\mathcal{S}\,\big|\{p_i,h_i,c_i\}\rangle_{i\in\text{in}} \\ &= g_{\text{YM}}^{N-2}\sum_{\ell}^{\infty} g_{\text{YM}}^{(\ell)}\,\mathcal{M}^{(\ell)}(\{p_i,h_i,c_i\})\end{aligned} \tag{1.17}$$

We can now study the behavior of these amplitudes when individual gluons become soft and their corresponding asymptotic charges. At the tree level, we have

$$\lim_{\omega_q\to 0}\mathcal{M}_{N+1}^{(0)}(\{p_i,h_i,c_i\}) = g_{\text{YM}}\,S^{(0)}(\{p_i\},q,\varepsilon_q)\mathcal{M}_N^{(0)}(\{p_i,h_i,c_i\}) \tag{1.18}$$

where, the leading gluon soft factor upon $q\to 0$ limit becomes

$$S^{(0)}(\{p_i\},q,\varepsilon_q) = \sum_{i\in\text{out}}\frac{p_i.\varepsilon(q)}{p_i.q}t_i^c - \sum_{i\in\text{in}}\frac{p_i.\varepsilon(q)}{p_i.q}t_i^c.$$

The *soft* (zero-momentum) gluon behavior of this Yang-Mills amplitude in the infrared limit gives us infinite non-abelian charge conservation at the boundary, which are related to soft gluon theorems via Ward identities [62, 71–73].

The soft photon in our scattering matrix in 4*D* is equivalent to insertions of a $U(1)$ Kac-Moody current on the S^2 at the boundary. The current on the 2-sphere of null infinities generates the $U(1)$ Kac-Moody Current algebra, and the same for the 4*D* non-abelian gauge theory with gauge group $\mathcal{G}$ is called $\mathcal{G}$ Kac-Moody Current algebra [2, 74]. The soft gluon theorems are the Ward identities associated with the Kac-Moody symmetries, which are the asymptotic symmetries of non-Abelian gauge theories.

QCD in particular:

In the case of QCD, the coupling decreases at high energies. We have the coupling,

$$g_{YM}^2(k^2) = \frac{1}{\beta_0\ln(k^2/\Lambda)},$$

where β_0 is a constant, and k is the energy of the process. Here, Λ is the scale at which the coupling becomes strong as low energy.

One can see that due to the logarithmic nature, the coupling increases with decreasing

dressing factors comprise the soft gluons, which cancel the IR divergences.

energy ($k^2 \to 0$). This means that the coupling becomes large at low energies. As a result of this, we can't rely on the perturbation theory. However, one can opt for perturbative analysis once we have our energy $k^2 > \Lambda$. This makes the coupling g^2_{YM} smaller. This implies our perturbative energy scale is defined below $1/\Lambda$.

Hence, in pure Yang-Mills or non-abelian gauge theory, when we talk about the soft particles, we refer to the energy of the soft particle as very less compared to the energy scale of the theory. The above analysis validates our perturbative analysis in non-abelian gauge theory.

In the next section, I will explain gravitational scattering in particular.

1.2.2 Scattering $\mathcal{S}$-Matrix in Gravity

$\mathcal{S}$-Matrix in Gravity are the observables in Quantum Gravity (QG) [75–80]. One important consideration from the aspects of QFT is to find the amplitudes to be gauge-invariant and can be calculated in terms of correlation functions.

We are more focused on the scattering processes involving massless particles because of our interest in gravitational wave emissions. Unitary and Lorentz invariant Scattering $\mathcal{S}$-matrix involving massless particles impose strong constraints on the gravitational dynamics. Taking the motivation from the string scattering amplitudes, Veneziano et al. studied the high energy behavior of the massless graviton scattering amplitudes [81]. Recently, these scattering processes have been highlighted because of the discoveries of gravitational waves by the LIGO-VIRGO-KAGRA (LVK) and IndiGO Collaborations [82–85]. Recent state-of-the-art calculations show applications in the classical scattering processes from the quantum scattering amplitude methods[12] [86–94]. Relevant gravitational scattering amplitudes have been studied through the $\mathcal{S}$-matrix bootstrap program using BCFW recursion [95–97] and BG recursion algorithms [98–103] for tree amplitudes, which are extended to loop level using unitary methods [104].

The scattering matrix element describing the process of scattering of n in coming and $N-n$ outgoing particles is given by

$$\mathcal{S}_{fi} = \langle p_1, p_2, \cdots p_n | p_{n+1}, \cdots p_N \rangle = i(2\pi)^4 \delta^{(4)} \Big(\sum_{i=1}^{N} p_i \Big) \mathcal{M}(p_i, h_i, s_i). \tag{1.19}$$

Here, $\mathcal{M}$ is our Lorentz invariant scattering amplitudes, which is a function of momentum (p_i), helicity (h_i), and any internal quantum number (s_i) of the particles. This transition satisfied the

[12] Double Copy relations reviewed in section 1.5.

crossing symmetries[13] between the incoming and outgoing particles. Hence, we can assume all the momenta p_i, $i = 1, \cdots, N$ are to be unidirectional (either incoming or outgoing based on your convenience). Hence, the conservation of momenta becomes[14], $\sum_{i=1}^{N} P_i = 0$.

This probability amplitude for particles scattering off one another in the presence of a gravitational potential plays a crucial role in understanding gravitational interactions. In short, scattering $\mathcal{S}-$matrix in gravity acts as a bridge between classical and quantum gravity, helping scientists to understand our universe at a fundamental level.

Now, in this regard, we think classical observers might need some help. So,

How can a quantum observer help a classical observer being at infinities?

The answer could be *Universalities*! Yes, universalities of the high energy scattering processes have been studied and applied for gravitational wave observations.

1.2.3 Universalities of Scattering Amplitudes

In general, every symmetry leads to constraints on physical observables, such as the scattering amplitudes. We will discuss the two universal features of scattering amplitudes, which come in the Infrared and collinear regimes. These are the two common sources of divergences in scattering studies.

Collinear divergences are important concepts that arise in the process involving massless particles when the angle between the momenta goes to zero. These divergences can be factorized, which means the divergent part of the amplitudes can be separated from the finite part. In the language of CFT, this limit typically refers to the situation where two operators approach each other along a specific direction on the conformal sphere. I will highlight this later in the chapters.

Soft divergences arise when the energy of a massless particle becomes small. The amplitude becomes infinite due to the long-range nature of the interactions mediated by these infinite numbers of soft massless particles. For classical scattering, this ultra-relativistic limit of the

[13]The scattering amplitude is a complex function of the momenta of particles. This function can be analytically continued to the other regions of our momentum space. When we analytically continue to the region where the momentum of a particle changes sign, we can get a relation to our initial amplitude. The change of sign of the momenta physically means treating the particle as the antiparticle. This is crossing symmetry.

[14]This can be considered for a scattering process of N massless particles with $p_i^2 = 0$ in the Spinor-Helicity (SH) basis as $\sum_{i=1}^{N} \langle li \rangle [im] = 0$. For reference, one can see the review in 1.2.4. The scattering amplitude $\mathcal{M}$ can be written in the SH basis as, $\mathcal{M}(\lambda_i, \bar{\lambda}_i, s_i)$.

scattering helps us to find observables in relation to soft graviton theorems in gravity[15] and supergravity theories[16] [107–112]. Let us discuss these relationships in a bit of detail.

Soft Factorization:

The *soft* limit of the amplitude is defined by taking the momenta of one or more external particles to zero[17]. Quite generally, under the *soft* limit, the amplitude factorizes into a universal (*soft*) factor, which contains the divergent part of the amplitude times the amplitude without the soft particle(s) insertions. This factorization is known as the *soft theorem*.

We have any $n-$point gravity scattering amplitude with one soft particle,

$$\mathcal{M}_n(\cdots,a,s,b,\cdots) \overset{p_s\to 0}{\longrightarrow} \mathrm{Soft}^{\mathrm{Gravity}}(a,s,b)\,\mathcal{M}_{n-1}(\cdots,a,b,\cdots). \tag{1.20}$$

where p_s is the momenta of the *soft* field and a,b are the adjacent fields[18]. I will explain this explicitly in the case of pure Yang-Mills (YM) theory as an example later in chapter 2. At the tree level, the soft factor is given by

$$\mathrm{Soft}^{\mathrm{Gravity}}(a,s,b) = \frac{1}{\varepsilon^3}\mathrm{Soft}(0)^{\mathrm{Gravity}}(a,s,b) + \frac{1}{\varepsilon^2}\mathrm{Soft}(1)^{\mathrm{Gravity}}(a,s,b) + \frac{1}{\varepsilon}\mathrm{Soft}(2)^{\mathrm{Gravity}}(a,s,b) \tag{1.21}$$

where ε is the soft momentum parameterization, which means $\varepsilon \to 0$ is our soft limit [113, 114]. The label (0), (1), and (2) indicate the leading, subleading, and sub-subleading universal *soft* terms for tree-level gravity amplitudes. Similarly, for gauge theories, like (Super)Yang-Mills up to subleading, we have the universal soft factors at tree level,

$$\mathrm{Soft}^{(\mathrm{S})\mathrm{YM}}(a,s,b) = \frac{1}{\varepsilon^2}\mathrm{Soft}(0)^{(\mathrm{S})\mathrm{YM}}(a,s,b) + \frac{1}{\varepsilon}\mathrm{Soft}(1)^{(\mathrm{S})\mathrm{YM}}(a,s,b).$$

Soft limits of amplitudes at the tree level provide important new insights about the symmetries of certain theories [47, 49, 50, 115, 116]. For example, as I explained in the previous sections,

[15] Of course, these universalities are true in gauge theories also.

[16] Especially maximally supersymmetric theories like $\mathcal{N}=4$ super Yang-Mills and $\mathcal{N}=8$ supergravity is our natural laboratory for some classical analysis. The simplicity comes from the potential UV finiteness at all loop orders [105] in supergravity. High symmetries in the theories lead to exact analytic solutions for various physical quantities (like precise checks in $\mathcal{N}=4$ SYM), help in understanding the gauge/gravity dualities, multi-loop calculations of scattering amplitudes, etc.[106].

[17] The particle has to be massless for such a limit to make sense.

[18] When a virtual particle goes on-shell, there will be two possibilities, for the soft particle to get attached to. One to each of the adjacent external legs. Hence, after taking the soft limit, the soft factors will have information of these two adjacent external legs. This is shown in figure 2.3 in section 2.4 of chapter 2 with an example in pure YM theory.

soft gluon theorem in Yang-Mills theory is related to *large* gauge transformations and soft graviton theorem in Einstein's gravity is related to the so-called Bondi-Metzner-Sachs (BMS) symmetries [8–11]. I will discuss the infinite symmetries of the asymptotics in detail in the upcoming sections.

This universal factorization can be extended to subleading order in electromagnetism and to sub-subleading order in gravity [12, 43–46, 48, 55, 57, 117–121]. Thus, studying the soft limits of amplitudes, even at the tree level, can teach us more about the symmetries of the theory.

Collinear Factorization:

Another important limit of amplitudes is the collinear limit, which is when we have the momenta of two massless external particles are taken to be collinear. Again, the amplitude factorizes into a collinear factor containing the divergence times the amplitude with the collinear particles replaced by another particle [89, 103, 122–125].

In the collinear limit, we take the momenta of two adjacent particles p_1 and p_2 to be collinear. Under this limit, the two particles can fuse to give another particle with momentum $p_{12} = p_1 + p_2$. We parametrize the momenta of the collinear massless particles,

$$p_1 = x\, p_{12}, \quad p_2 = (1-x)\, p_{12},$$

where x corresponds to the combined momentum p_{12}. Since $p_1 + p_2 = p_{12}$, we see that, for massless fields, the collinear limit $p_1 || p_2$ implies $p_1 \cdot p_2 \propto p_1^2 = 0$ which is equivalent to the condition $p_{12}^2 \to 0$. We have the Collinear amplitude as,

$$\mathcal{M}_n(1^{h_1}, 2^{h_2}, \ldots, n) \xrightarrow{1||2} \sum_h \text{Split}_{-h}^{\text{Gravity}}(z, 1^{h_1}, 2^{h_2}) \mathcal{M}_{n-1}(p^h, \ldots, n), \tag{1.22}$$

where the above *split* factor has all the collinear divergences of the scattering process having all the information of the two collinear particles [126].

I will thoroughly review these universal factorizations of amplitudes, with examples in the upcoming chapters, especially being focused in *celestial* basis considering our application in Celestial Holography. In this context, these two limits of the scattering amplitudes are the basic *ingredients* for constructing *celestial* scattering amplitudes.

In the next subsection, I will introduce the Spinor-Helicity Formalism, which is considered to be helpful and heavily used in our analysis of scattering amplitudes and symmetry studies.

1.2.4 A brief review of Spinor-Helicity (SH) Formalism

The helicity spinors are left and right-handed representations of the Lorentz group $\mathrm{SO}(1,3) \sim \mathrm{SL}(2,\mathbb{C})$. We denote the left and right-handed helicity spinors by h_α and $\tilde{h}^{\dot\alpha}$ respectively. Lorentz invariant contractions of spinors is defined using the completely antisymmetric rank 2 tensor $\varepsilon^{\alpha\beta}$ defined as

$$\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = -\varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.23}$$

The contractions are then defined as

$$\begin{aligned} \langle\lambda\chi\rangle &\equiv \varepsilon^{\alpha\beta}\lambda_\alpha\chi_\beta = \lambda_\alpha\chi^\alpha = -\lambda^\alpha\chi_\alpha = -\langle\chi\lambda\rangle \\ [\lambda\chi] &\equiv \varepsilon_{\dot\alpha\dot\beta}\widetilde{\lambda}^{\dot\alpha}\widetilde{\chi}^{\dot\beta} = \widetilde{\lambda}^{\dot\alpha}\widetilde{\chi}_{\dot\alpha} = -\widetilde{\lambda}_{\dot\alpha}\widetilde{\chi}^{\dot\alpha} = -[\chi\lambda]. \end{aligned} \tag{1.24}$$

Wherever we have angular brackets, we understand that it is the contraction of the left-handed spinor, whereas the square bracket is the contraction of the right-handed spinor. We thus suggestively denote left handed spinor by $|\lambda\rangle^\alpha$ and right handed spinor by $[\lambda|^{\dot\alpha}$. A given null momentum p^μ can be written as a bispinor

$$p^{\alpha\dot\alpha} = \sigma_\mu^{\alpha\dot\alpha} p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \equiv |p\rangle[p| \tag{1.25}$$

where $\sigma_\mu = (1, \sigma_x, \sigma_y, \sigma_z)$ and $|p\rangle, [p|$ are some spinors. For real physical momentum, the two spinors and their contractions are related by complex conjugation $([p|)^* = |p\rangle)$ and $\langle pq\rangle^* = [qp]$. Given the bispinor of a 4-vector p^μ, we can recover the 4-vector as follows:

$$p^\mu = \frac{1}{2}\sigma^{\mu\alpha\dot\alpha} p_{\dot\alpha\alpha} = \frac{1}{2}\bar\sigma^\mu_{\dot\alpha\alpha} p^{\alpha\dot\alpha},$$

where $\bar{\sigma}_\mu = (1, -\sigma_x, -\sigma_y, -\sigma_z)$. The inner product of two null momenta $p^\mu = |p\rangle[p|$ and $q^\mu = |q\rangle[q|$ is given in terms of spinor contractions as

$$p \cdot q = \frac{1}{2}[pq]\langle qp\rangle. \tag{1.26}$$

If we have several momenta, which is usually the case in scattering processes, say $p_1, \ldots, p_n$, then we shorten the notations further and denote the corresponding spinors by $|i\rangle, [i|$ for $i = 1, \ldots, n$. The momentum conservation can then be expressed as

$$\sum_{j=1}^{n} \langle ij\rangle[ji] = 0 \tag{1.27}$$

for $p_i = |i\rangle[i|$. i be any one out of the n external momenta. One can also express polarisations in terms of spinors but we will not need it explicitly in our discussions.

Now, let us discuss the symmetries of the Asymptotically flat spacetimes.

1.3 Asymptotic Symmetries in Gravity

It has been observed that in gauge and gravity theories, there is an enhancement of symmetry at the boundaries. For asymptotic boundaries, such enhanced symmetries are known as the asymptotic symmetries. Moreover, a deeper understanding of these symmetries might help in understanding the black hole microscopics [127–131]. This necessitates the computation and analysis of asymptotic symmetries.

The usual method of finding asymptotic symmetry of a theory is governed by finding symmetry transformation parameters for various fields that preserve their falloff conditions at the boundary. For a theory of gravity, we look for asymptotic isometry transformations that leave the boundary falloffs of various gravitational fields intact. The falloff conditions are determined with respect to the asymptotic geometry. In addition to fall-offs in Eq.(1.9), our search for Asymptotically flat spacetime is towards finding the most general diffeomorphisms preserving Bondi gauge conditions in Eq.(1.5).

1.3.1 Bondi–van der Burg–Metzner–Sachs (BMS) at Infinites

In one of his works in the line of understanding symmetries [5], R. Sachs mentioned the requirement of Asymptotically Flat Spacetimes (AFS) to understand inhomogeneous Lorentz transformation symmetry (Poincaré symmetry) as the *approximate* symmetry, which breaks in the presence of dynamical gravitational fields. Quantum mechanically, these approximate symmetries become accurate. For this kind of AFS, the work of Bondi, Metzner et al. [8–10] implies that, with the appropriate boundary conditions, we can obtain a symmetry enhancement in the presence of gravitational fields. The corresponding symmetry group at that time was named the generalized Bondi-Metzner group, which was later translated as the Bondi–van der Burg–Metzner–Sachs (BMS) group [11].

BMS is the semi-direct product of the Lorentz Transformation (LT) group acting on an infinite dimensional abelian group of supertranslations. We have our well-known global Poincaré transformations in the bulk *enhanced* to global BMS at the boundaries. This can be written symbolically as,

$$\begin{cases} \text{Poincaré Group} = \text{ Lorentz Group } \ltimes \text{ Translation} \\ \qquad\qquad\qquad\qquad \downarrow \\ \quad \text{Global BMS} = \text{ Lorentz Group } \ltimes \text{ Supertranslation} \end{cases}$$

- **Lorentz Transformation (LT):** We have the isomorphism, $\mathrm{SO}(3,1) \equiv \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. Hence, this becomes a two-dimensional Conformal transformation (Möbius Transformation) at the asymptotic two-sphere. Hence, one can express the LT in terms of the $\mathrm{SL}(2,\mathbb{C})$ matrices, which act on the asymptotic coordinates of our AFS spacetime. Hence,

$$(z,\bar{z}) \longmapsto \left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}). \tag{1.28}$$

- **Supertranslation:** At $\mathcal{I}^{\pm}$ $(\mathbb{R}\times\mathbb{S}^2)$, we have an infinite dimensional translation of the advance or retarded coordinates. This transformation is parameterized by any angle-dependent function on the sphere and is popularly called *supertranslation* transformations. Hence, we have an infinite of translations,

$$u \to u + f(z,\bar{z}),$$

  where, $u = t - r$ are the retarded time, and $(z,\bar{z})$ are the stereographic coordinates on the sphere of $\mathcal{I}^+$. When $f(z,\bar{z})$ = constant, we have u-translation generated at the boundary. $f(z,\bar{z}) = Y_1^{(-1,0,+1)}$, harmonics on $\mathcal{CS}^2$ generates *three* spatial translation for $\ell = 1$ harmonics and one time translation for $\ell = 0$ harmonic. These global translation killing vector fields $\xi(f)$ on the $\ell = 0$ and $\ell = 1$ spherical harmonics preserve the Bondi gauge and the metric fall-offs at $r \to \infty$. Any generic function allows separate translations along every null generator of $\mathcal{I}^+$.

Here is a little demonstration of the *broken* supertranslation symmetry [12, 15, 132] at the boundaries of our AFSs. The Lie action on the boundary data $\{N_{zz}, m_B, C_{zz}\}$ are given by,

$$\begin{cases} \mathcal{L}_f N_{zz} = f\partial_u N_{zz}, \\ \mathcal{L}_f m_B = f\partial_u m_B + \dfrac{1}{4}\left[N^{zz}D_z^2 f + 2D_z N^{zz} D_z f + c.c.\right] \\ \mathcal{L}_f C_{zz} = f\partial_u C_{zz} - 2D_z^2 f. \end{cases} \tag{1.29}$$

All the quantities in the above are defined in section (1.1.1).

A brief remark:

Let us define the Mink$_4$ vacuum as $\Big|\{N_{zz} = 0, m_B = 0, C_{zz} = 0\}\Big\rangle$. The Lie derivative action along the supertranslation killing vector on the defined Minkowski vacuum from Eq.(1.29) implies,

$$\mathcal{L}_f C_{zz} \neq 0. \tag{1.30}$$

This means our vacuum is *supertranslated*. One can notice here that this supertranslated spacetime has $m_B = 0$ and $N_{zz} = 0$. This is consistent with the fact that the physical mass of our spacetime remains invariant under diffeomorphism transformations. However, we have non-zero C_{zz} and to make this vanish, we have to take $C_{zz} = -2D_z^2 C$, where $C(z,\bar{z})$ parametrizes classically *inequivalent* vacua[19] [11]. Hence, Eq.(1.30) can be written as,

$$\mathcal{L}_f C = f, \tag{1.31}$$

which implies $C \to C + f$ under supertranslation[20].

One can find the asymptotic supertranslation charges corresponding to this symmetry,

[19] C is the Goldstone boson for *broken* supertranslation symmetry.

[20] Note: For $f(z,\bar{z}) = Y_\ell^m$, for $\ell = 0, 1$ the action of the covariant derivative vanishes. Hence, the global spacetime translation is not broken at the boundaries of our Mink$_4$.

which is defined as [11],

$$Q_f^+ = \frac{1}{4\pi G}\int_{\mathcal{I}_-^+} d^2z\, \gamma_{z\bar{z}}\, f\, m_B. \tag{1.32}$$

This is the surface charge at $\mathcal{I}_-^+$. Similarly, we can have the expression for Q_f^- on $\mathcal{I}_+^-$. Readers can find the explicit derivations to the charge expression in the recent work by Andrew Strominger in [11].

Extended BMS (e-BMS)

Now we understand that the global BMS corresponds to the six global Lorentz rotational generators with the supertranslation on the *celestial* sphere. So the question comes,

What if we have a generic vector field parametrization at every angle on the $\mathcal{CS}^2$?

By this, we mean the global conformal transformation in Eq.(1.28) can be localized to local transformations, which preserves the asymptotic boundary metric infinitesimally. This local transformations are called *superrotaions* on the boundary sphere [133, 134]. Hence, we have the infinite-dimensional semi-direct product of *extended* BMS (or e-BMS) as,

$$\text{local e-BMS} = \text{Superrotations} \ltimes \text{Supertranslations.}$$

Let's say that the vector parameterization is $X^A(z,\bar{z})$; then, similar to supertranslation, we can write the superrotation charge on the boundary sphere as [11],

$$Q_X^+ = \frac{1}{8\pi G}\int_{\mathcal{I}_-^+} d^2z\, [X_{\bar{z}}\, N_z + X_z N_{\bar{z}}]\,. \tag{1.33}$$

BMS_4 symmetry is physically relevant for our gravitational systems in four spacetime dimensions[135]. The developments of this infinite symmetry and its application in conformally invariant systems and relation to two-dimensional CFT (CFT_2) is initially stated in the seminal work [136].

The Lie algebra $\mathfrak{bms}_4$ is defined as the semi-direct sum of the Lie algebra of conformal killing vectors of the Riemann(celestial) sphere corresponding to infinitesimal local conformal transformations with supertranslation generators on the Riemann sphere [133]. Barnich *et. al.* provided the detailed derivation of the generalization of $\mathfrak{bms}_4$ algebra[21] along with proof

[21] and $\mathfrak{bms}_3$. In three spacetime dimensions, the asymptotic symmetry group is BMS_3, where the diffeomor-

that the asymptotic symmetry algebra in four dimensions is represented by spacetime vectors, which can be generalized to the extended algebra via conformal rescaling of the boundary metric [134]. The surface charges associated with asymptotic symmetries in $4D$ flat spacetimes at null infinity and their transformation properties are constructed. The divergence and non-divergence of the supertranslation and superrotation charges in the case of the Kerr black hole have been studied [13].

Supersymmetric Extended BMS (SUSY-e-BMS)

BMS is a bosonic asymptotic symmetry transformation. However, one can be curious about the supersymmetric extension of this symmetry group in supersymmetric gauge and gravity theory [137–139]. In this scenario, the entire BMS analysis can be done for the fermionic theories, where we have the super-Lorentz generators as functions of the angular coordinates on the sphere. The symmetry group will be the supersymmetric version of extended BMS symmetry transformation, which we can call SUSY-e-BMS.

In part of this thesis work, I will dedicate a chapter towards understanding this as an application of Celestial Holography followed by some recent developments by Taylor et al.[7].

1.4 Techniques to Find Asymptotic Symmetries

This small review intends to highlight some of the techniques used in the literature to compute the symmetries of asymptotically flat spacetimes. These are some generalized techniques specific to spacetimes with asymptotic boundaries.

1.4.1 Covariant Phase Space (CPS) formalism

The study of asymptotic symmetries in gauge and gravity theories leads us to find the asymptotic surface charges at the boundary of the spacetimes. One of the traditional techniques was developed by Wald and Zoupos [140] in the context of Covariant Phase Space (CPS) constructions and defining the surface change using the symplectic structure. This study is most important in the case of radiative asymptotically flat spacetimes. Using this prescription, we construct the complete charge algebra, which contains all the information on the gravitational radiation flux at the null boundaries of the spacetimes. This is a generalization of phase space in classical mechanics to classical covariant field theories.

Our literature on asymptotic symmetries computation is rich in using this language. Al-

phism on the circle will be an infinite-dimensional extension of the Lorentz group.

though we have not used this formalism in this thesis work, I want to give a brief review of the methodology below for the readers to get a feel of this years-old formalism. For in-depth knowledge about formalism, one can find the appropriate references herein.

Methodology:

- In the asymptotics, conserved surface charges imply symmetry constraints. We must construct the physical phase space Γ by imposing certain gauge conditions.

  Let us consider an asymptotic field as gravitational radiation information in terms of the field C_{AB}, and the charge corresponding to the infinitesimal *supertranslation* symmetry given by Q_f in section 1.3.1, where $f(z,\bar{z})$ being the parametrized function on the celestial sphere. Using the Dirac bracket prescription as an example, we have

  $$\delta C_{AB} = \{Q_f, C_{AB}\}.$$

  Similarly, we can have all the relations for all possible gravitational radiation data and possible symmetries at the null boundaries of $4D$ Asymptotically flat Minkowski spacetimes [141].

- **Symplectic form and charges**: Let's consider the linearised fluctuation around a static background, $g_{\mu\nu}$: $\delta g_{\mu\nu} = h_{\mu\nu}$, which solves the linearised vacuum Einstein's equation. This, in other words, means that the variations are tangent to the phase space of solutions. Now, we can define the pre-symplectic form[22] corresponding to two on-shell field variations (h_1, h_2) as the integral over the pre-symplectic current density one-form, $J(h_1, h_2)$ on a Cauchy hypersurface Σ, which solves $d*J = 0$. Hence,

  $$\omega(h_1, h_2) = \int_{\Sigma} *J(h_1, h_2). \tag{1.34}$$

  Suppose we have diffeomorphism ξ, which acts non-trivially on the phase space. With proper choice of gauge conditions and constraints, we can make ω invertible, which fixes to give the symplectic form $\Phi(h_1^{\xi}, h_2)$. The conserved surface charge associated with this symmetry is given by the symplectic form as follows,

  $$Q^{\xi}_{\partial\Sigma} \equiv \Phi(h_1^{\xi}, h_2) = -\frac{1}{16\pi}\int_{\partial\Sigma} *F \tag{1.35}$$

[22] This is well explained by Wald in his work of local constraints and symmetries:[142].

Where F is the gravitational field strength two-form such that $*J = \frac{1}{16\pi} d * F$ [141]. This charge infinitely generates the symmetry acting on the phase space using the Dirac bracket prescription:

$$\{Q^{\xi}, h(u,z,\bar{z})\} = \delta_{\xi} h(u,z,\bar{z}) = h^{\xi}(u,z,\bar{z}). \tag{1.36}$$

Hence, to have a suitable asymptotic theory, we need to construct the phase space with the given symplectic pair $(Q_{\xi}, \delta_{\xi} g_{\mu\nu})$ at the boundary of the manifold.

I request the interested reader to look for this great thesis review by Adrien Fiorucci [143] for more details on this CPS formalism in case of spacetimes with non-trivial background curvature.

1.4.2 Hamiltonian Formulation

Another important technique in this direction is the Hamiltonian formulation of classical field theories for finding asymptotic symmetries [144–150]. Henneaux and Troessaert uncovered and simplified the asymptotic symmetry analysis at the spatial infinities [151–153]. Later, these techniques were used in constructing the phase space at null infinities [154–157].

The advantage of this technique over other techniques is that it helps in deriving asymptotic symmetries from the first principle. This requires a well-defined asymptotic phase space, a symplectic phase space, a symplectic form defined on this phase space, a Hamiltonian, and an action of the Global symmetry group on the defined phase space. Here is a brief summary of the methodology used for symmetry analysis.

Methodology:

- Gauge transformations are symmetry transformations that involve arbitrary functions of spacetime points. These transformations put constraints on the spacetime described by Dirac Constraints [158]. The symmetry generators corresponding to this symmetry are functions of the gauge parameter and the constraints, along with surface terms representing the local transformations on the sphere at the spacetime boundary.

- Asymptotic symmetries are canonical gauge transformations that preserve the boundary conditions, and this infinitesimal gauge transformation is a Hamiltonian vector field that has a canonical generator. This implies it preserves the symplectic structure [159–161].

- The Dirac brackets between two canonical asymptotic generators generate another asymptotic symmetry transformation at the boundary, which depends on the boundary surface term. If this surface term vanishes, we call those transformations *trivial/small/proper* gauge transformations. If not, we have the *non-trivial/large/improper* gauge transformations. So *small* gauge transformations form an ideal of the asymptotic symmetry algebra for all of the asymptotic symmetries.

- We know that physical states are trivial representations of the Poincaré group. In the asymptotics, the generators of the *improper/large* gauge transformations are the true physical observables.

Literature in Asymptotic symmetries is primarily based on this formalism; this technique is tedious. However, recently, some developments in flat space holography have helped us understand this more simply. In the next section, I will briefly introduce this new Celestial holographic technique and request the readers to follow the next chapter 2 of this work for a complete review of the technique.

1.4.3 Holographic Techniques

A simple yet deeply mystical concept of our Universe as a Hologram!

Bekenstein and Hawking's formula [162, 163] for black hole entropy hints towards the *holographic* universe. This formula signifies that all the information (quantum nature) of a black hole is encoded using the holographic correspondence at the boundary, which is the event horizon of the black hole.

The holographic principle is a concept proposed by Gerard 't Hooft and later by Leonard Susskind and Juan Maldacena, who played a pivotal role in the development of the principle. The most sophisticated description of the holographic principle [164–166] is in the case of AdS spacetime [167] and later for our type spacetimes with positive cosmological constants [168, 169]. Considering the success of AdS/CFT correspondence [166, 167], scientists are motivated towards flat space holography, which is to apply this correspondence for quantum gravity in asymptotically flat spacetimes [170–175].

To understand the nature of quantum gravity in flat spacetimes, we need the language of Conformal Field Theory (CFT), where the asymptotic symmetries put constraints on the dual CFT sitting at the conformal boundary (denoted by $\mathcal{I}^{\pm}, i^0,$ and $i^{\pm}$) of the flat spacetime. This

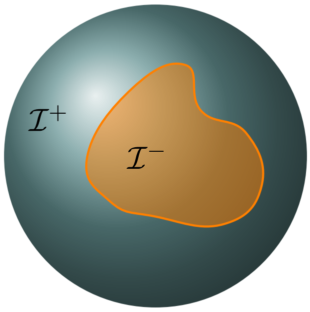


Figure 1.2: Projected null boundaries, $\mathcal{I}^{\pm}$ of Mink$_4$ on the boundary $\mathcal{CS}^2$.

holographic projection of our Mink$_4$ on the boundary sphere is shown in Fig.1.2.

Modernization of Flat Space Holography:

Here, I will list two of the most recent modern-day techniques used in the context of flat space holography.

- ***Celestial* Holography (CH):**

4D bulk of Mink$_4$ $\longrightarrow$ 2D sector of the Sky!

As the name suggests, this is related to the holographic projection of our 4D Mink$_4$ on the 2D sector of the boundary spacetime $\mathbb{R}\times\mathbb{S}^2$ or the *sky*. Hence, the name 'celestial' signifies the studies on the 2D sphere of the boundary called the *celestial* sphere[23] (which we denote by $\mathcal{CS}^2$). Starting from BMS [5, 8, 9, 176] on studies of asymptotic symmetries and Barnich and Troessaert's BMS/CFT [134], it is well known that the symmetry group of AFS is not Poincarè rather an infinite-dimensional enhancement of this. The later realization of Andrew Strominger related to the antipodal matching between the past and future null infinities helps us solve the gravitational scattering problems [15].

The *celestial* holographic principle [2, 11, 177–181] holds the idea of mapping the *in* coming and *out* going particles in the scattering to the conformal operators ($\mathcal{O}^{\pm}_{\Delta_i}(z_i,\bar{z}_i)$) at the boundary S^2. Here, the momentum p_i of the incoming/outgoing particle is interpreted in terms of the conformal dimension Δ, and the momentum direction depends on the point of insertions $(z,\bar{z})$ on the sphere. This implies mapping the wave functions in the bulk 4D Mink$_4$ to the boundary conformal primary wave functions. All this now implies the *exact* map of the scattering amplitudes to the conformal field correlators on our 2D

[23]The celestial sphere is the Riemann sphere on the boundary of the Minkowski space.

$\mathcal{CS}^2$. Generically, the map can be defined for a $D+2$ dimensional flat spacetime to a D dimensional *celestial* CFT (CCFT)[24] living on the S^D.

The collinear limit of amplitude turns into an operator product expansion (OPE) of conformal operators of the Celestial Conformal Field theory (CCFT) [182–185] as identical momentum directions correspond to the same operator insertion points on $\mathcal{CS}^2$. An interesting fact is that the soft and collinear limits of scattering amplitudes can be used to read off the asymptotic symmetries in the context of CCFT [184, 186]. It turns out that to calculate the asymptotic symmetries of a theory, we need to probe the universal soft and collinear sectors of the scattering amplitudes. This has been used to reproduce the BMS algebra in [186] and [187] for pure gravity and large gauge algebra for Einstein Yang-Mills theory [188]. Recently, it has also been used to compute the $\mathcal{N}=1$ supersymmetric extension of the BMS algebra [7].

In this thesis, I will be specific to this technique in particular to find the asymptotic symmetries. Readers can look for Chapter 2 for more details.

- ***Carrollian* Holography (CarrH):**

4D bulk of Mink$_4$ $\longrightarrow$ Full 3D Sky!

Contradicting the expectations of flat space holography to be a co-dimension one holography, CH proposes a co-dimension two holography for flat Mink$_4$. To address this with exact holography like in AdS/CFT, recent literature proposed another well-known perspective called *Carrollian* Holography (CarrH) [189–192]. One can call this to be a *modified* version of *Celestial* Holography[25].

This implies gravity in $D+2$ dimensional asymptotically flat spacetimes is dual to the *Carrollian* CFT (CarrCFT) in $D+1$ dimension sitting on the null boundary of the AFSs[26]. It is started from isomorphism at the algebraic level between the asymptotic symmetries of the flat spacetimes (BMS) in $D+2$ dimension (BMS$_{D+2}$) and non-Lorentzian CFT, the conformal Carrollian algebra in $D+1$ dimension (CCarr$_{(D+1)}$) [59, 196–198]. Initial

[24] In CCFT, one describes the four-dimensional physics in terms of the conformal correlators of two-dimensional CFT on the celestial sphere living at the null infinities of the Minkowski flat spacetime. The map from amplitudes in bulk to conformal correlators on the boundary is the Mellin transform.

[25] *Modified* as to represent the *modified* Mellin transformation explained in literature [193–195].

[26] Carrollian limit (ultra-relativistic limit) boosts a time-like boundary to a null boundary, and this maps the $D+1$ dimensional conformal Carroll algebra to $D+2$ dimensional Lorentz subalgebra of bulk Mink$_{D+2}$ [196].

works in this direction have been done in this direction for $3D$ flat spacetimes [197–205], then some recent development is in $4D$ to recover BMS_4 from $CCarr_3$ [190–195, 206].

Amongst all the advantages, one important consideration is given here [189] in the case of AdS spacetime to recover the flat space results in the *singular* limit (Flat space limit, AdS length $\to 0$). The recent constructions for computing certain Witten diagrams in terms of Carroll conformal correlation functions on the boundary help us develop an understanding of 'scattering' in AdS spacetimes.

I will highlight more on this in Chapter 6.

1.5 Relating Gauge and Gravity Theory

What gauge theory brings to gravity scattering amplitudes?

Recent investigations into gravity and gauge theory amplitudes have resulted in non-trivial relationships between the two [207]. Gravity tree-level amplitudes can be expressed in terms of sums of products of gauge theory tree-level amplitudes. This can be described by different double copy formalisms [208–211]. The relevant double copy formalism reviewed in [212] was originally formulated as a relation between open and closed string amplitudes [208]. The corresponding relation in the low energy effective theory gives a relation between gauge theory and gravity amplitudes. Thus, one can explicitly calculate soft and collinear limits of amplitudes in gravity using the corresponding results of gauge theory.

1.5.1 Double Copy : A Brief Review

Let us briefly review the double copy (DC) technique, which plays a crucial role in our analysis. It is a multiplicative bilinear operation to compute the amplitudes in one theory using amplitudes from other simpler theories. This is a method to express gravity tree-level amplitudes in terms of sums of products of gauge theory tree-level amplitudes [209, 213]. Schematically,

$$\mathcal{M}^{(Gauge)} \times \mathcal{M}^{(Gauge)} \sim \mathcal{M}^{(Gravity)} \tag{1.37}$$

There are three different double copy formalisms for tree-level amplitudes: KLT (named after Kawai, Lewellen, and Tye) [208], BCJ (named after Bern, Carrasco, and Johansson) [209] and CHY (named after Cachazo, He, and Yuan) [210, 211] formalism. We refer to [212] for a detailed review of these formalisms. Here, we restrict our discussion to the application of double copy to soft and collinear limits of gravity amplitudes in terms of soft and collinear limits of gauge theory amplitudes. This means to take the *soft* and *collinear* limits of both sides of Eq.(1.37).

This formalism has been mostly explored in different theories. However, I will be focused here on the case of supersymmetric theories. Hence, we ask if it is possible to relate the soft and collinear limits specifically in $\mathcal{N}=4$ SYM to soft and collinear limits in $\mathcal{N}=8$ supergravity. Indeed this can be done [214–217]. This study is motivated towards understanding the amplitudes of supergravity theory in detail. $\mathcal{N}=4$ supersymmetric Yang-Mills and $\mathcal{N}=8$ supergravity are maximally supersymmetric theories and are rich in symmetries. Due to enormous symmetries, one can compute higher and higher loop amplitudes and show that they are finite [105]. In fact people argue that these are one of the simplest quantum field theories [106]. One can then study the soft and collinear limits of amplitudes in these theories to learn more about the symmetries. The study of soft and collinear limits in $\mathcal{N}=4$ SYM has already been done [218–220] and the corresponding CCFT was studied in [221].

Double Copy and Soft Limit:

Similarly, one can take the soft limit of the double copy relation to relate the soft factors in gravity and gauge theories. Let us start with the universal soft behavior of the tree level n-gluon amplitude. The soft factor when the i-th particle is taken to be soft, for either helicity, is given by,

$$\begin{aligned}\mathcal{A}_n^{\text{tree}}\left(\ldots,a,\varepsilon i^{\pm},b,\ldots\right) \xrightarrow{\varepsilon\to 0} &\left(\frac{1}{\varepsilon^2}\mathcal{S}^{(0)}_{\text{Gauge}}(i,a,b)+\frac{1}{\varepsilon}\mathcal{S}^{(1)}_{\text{Gauge}}(i,a,b)+\mathcal{O}(1)\right)\\ &\times\mathcal{A}^{\text{tree}}_{n-1}(\ldots,a,b,\ldots)\end{aligned} \tag{1.38}$$

Here, the soft limit is parameterized by a factor $\varepsilon\to 0$, as described in the last section. The factors $\mathcal{S}^{(0)}_{\text{Gauge}}$ and $\mathcal{S}^{(1)}_{\text{Gauge}}$ contains the soft divergences to leading and subleading order in the gauge theory. Similarly, the gravity amplitude also has this universal soft behavior with i-th particle going soft and is given by,

$$\begin{aligned}\mathcal{M}_n^{\text{tree}}\left(\ldots,a,\varepsilon i^{\pm},b,\ldots\right) \xrightarrow{\varepsilon\to 0} &\left(\frac{1}{\varepsilon^3}\mathcal{S}^{(0)}_{\text{Gravity}}(i,a,b)+\frac{1}{\varepsilon^2}\mathcal{S}^{(1)}_{\text{Gravity}}(i,a,b)\right.\\ &\left.+\frac{1}{\varepsilon}\mathcal{S}^{(2)}_{\text{Gravity}}(i,a,b)+\mathcal{O}(1)\right)\mathcal{M}^{\text{tree}}_{n-1}(\ldots,a,b,\ldots)\end{aligned} \tag{1.39}$$

where $\mathcal{S}^{(0)}_{\text{Gravity}}$, $\mathcal{S}^{(1)}_{\text{Gravity}}$ and $\mathcal{S}^{(2)}_{\text{Gravity}}$ are leading, subleading and subsubleading soft factors in the gravity theory. Double copy relates these soft factors as follows [113, 209]

$$\begin{aligned}\frac{1}{\varepsilon^3}\mathcal{S}^{(0)}_{\text{Gravity}}(s,n,1)+\frac{1}{\varepsilon^2}\mathcal{S}^{(1)}_{\text{Gravity}}(s,n,1)+\frac{1}{\varepsilon}\mathcal{S}^{(2)}_{\text{Gravity}}(s,n,1)\\ =\sum_{j=1}^{n}K^2_{sj}\left(\frac{1}{\varepsilon^2}\mathcal{S}^{(0)}_{\text{Gauge}}(j,s,n)+\frac{1}{2\varepsilon}\mathcal{S}^{(1)}_{\text{Gauge}}(j,s,n)\right)^2\end{aligned} \tag{1.40}$$

where $K^2_{sj}=\varepsilon\langle sj\rangle[sj]$.

Double Copy and Collinear Limit:

The KLT double copy was originally discovered in string theory as a relation between open and closed string amplitudes. Once the large string tension limit (also called the field theory limit) is taken, the KLT relation turns into a relation between gravity and gauge theory tree level amplitudes [216]. The general KLT relation for a general gravity tree level amplitude $\mathcal{M}^{\text{tree}}_n(1,2,\ldots,n)$ with n external legs (we have assumed n to be even below, but the odd case can also be written in a similar way with appropriate modifications) with color-ordered[27] gauge theory tree level amplitude $A^{\text{tree}}_n(1,2,\ldots,n)$ is given by [216].

$$\begin{aligned}\mathcal{M}^{\text{tree}}_n(1,2,\ldots,n)=&\,i(-1)^{n+1}A^{\text{tree}}_n(1,2,\ldots,n)\\ &\times\sum_{\substack{\sigma\in S_{n/2-1}\\ \tau\in S_{n/2-2}}}f(\sigma(1),\ldots,\sigma(n/2-1))\,\bar{f}(\tau(n/2+1),\ldots,\tau(n-2))\\ &\times A^{\text{tree}}_n(\sigma(1),\ldots,\sigma(n/2-1),1,n-1,\tau(n/2+1),\ldots,\tau(n-2),n)\\ &+\text{Permutations of }(2,\ldots,n-2).\end{aligned} \tag{1.41}$$

[27] Color-ordered or "color-stripped" amplitudes don't depend on the gauge group indices. These *partial* amplitudes are gauge invariant and depend on the kinematic invariants only. For instance, an N-gluon amplitude can be written as [122],

$$\mathcal{M}(\{p_i,h_i,c_i\})=\sum_{\sigma\in S_{N-1}}2^{N/2}\,\text{Tr}\{T^{c_1}T^{c_{2\sigma}}\cdots T^{c_{N\sigma}}\}A(1^{h_1},2^{h_{2\sigma}}_{\sigma},\cdots,N^{h_{N\sigma}}_{\sigma}).$$

Here, $A(1^{h_1},2^{h_{2\sigma}}_{\sigma},\cdots,N^{h_{N\sigma}}_{\sigma})$ is our color-ordeed or *partial* amplitude. The sum is over the $(N-1)!$ permutations of σ.

The functions f and $\bar{f}$ are defined as

$$\begin{aligned} f\left(i_1,\ldots,i_j\right) &= s\left(1,i_j\right)\prod_{m=1}^{j-1}\left(s\left(1,i_m\right)+\sum_{k=m+1}^{j} g\left(i_m,i_k\right)\right) \\ \bar{f}\left(l_1,\ldots,l_{j'}\right) &= s\left(l_1,n-1\right)\prod_{m=2}^{j'}\left(s\left(l_m,n-1\right)+\sum_{k=1}^{m-1} g\left(l_k,l_m\right)\right) \end{aligned} \tag{1.42}$$

where

$$g(i,j) = \begin{cases} s(i,j) := s_{ij} := \langle ij\rangle[ji], & i>j \\ 0, & \text{otherwise.} \end{cases} \tag{1.43}$$

Thus, every gravity state j on the LHS can be interpreted as the tensor product of the two gauge theory state on the RHS.

One can take the collinear limit on both sides of the KLT relation (1.41) to obtain a relation between the split factor for collinear states in gravity and the split factors in gauge theory. We describe this relation below. The collinear limit in gravity is written as [217]

$$\mathcal{M}_n^{\text{tree}}\left(1^{h_1},2^{h_2},\ldots,n\right) \xrightarrow{1\|2} \sum_{h=\pm} \text{Split}_{-h}^{\text{gravity}}\left(z,1^{h_1},2^{h_2}\right)\times M_{n-1}^{\text{tree}}\left(P^h,3,\ldots,n\right). \tag{1.44}$$

Using the KLT relation, the gravity split factor can be related to the “square" of gauge split factors as [216],

$$\begin{aligned} \text{Split}_{-(h+\tilde{h})}^{\text{gravity}}\left(z,1^{h_1+\tilde{h}_1},2^{h_2+\tilde{h}_2}\right) &= -s_{12}\times \text{Split}_{-h}^{\text{gauge}}\left(z,1^{h_1},2^{h_2}\right) \\ &\qquad \times \text{Split}_{-\tilde{h}}^{\text{gauge}}\left(z,2^{\tilde{h}_2},1^{\tilde{h}_1}\right). \end{aligned} \tag{1.45}$$

Here a state $h+\tilde{h}$ in gravity theory is written as the product of states $h,\tilde{h}$ in the two gauge theories and $s_{12} = \langle 12\rangle\,[21]$.

For instance, $\mathcal{N}=8$ supergravity amplitude can be related to the amplitudes in $\mathcal{N}=4$ super Yang-Mills in this way, and this leads to the relation,

$$\mathcal{N}=8 \;\text{ Supergravity} \sim (\mathcal{N}=4 \;\text{ Super Yang-Mills}) \otimes (\mathcal{N}=4 \;\text{ Super Yang-Mills}).$$

Note that the doubling of supersymmetry in this double-copy relation can be understood by counting the degrees of freedom on the two sides. Indeed, $\mathcal{N}=8$ supergravity has 256 states, which is twice the 128 states in $\mathcal{N}=4$ SYM. We will explain the explicit factorization of states for the case of $\mathcal{N} = 8$ supergravity into $\mathcal{N} = 4$ super Yang-Mills states in Chapter 5.

1.6 Flat space observer in AdS

Any physical information of a theory is encapsulated in its $\mathcal{S}$-matrix, which gives us the scattering amplitudes of an in-state in the far past to time evolve to an out-state in the far future. $\mathcal{S}$-matrix is a well-defined observable for quantum field theories in flat space-time. However, for theories in AdS space-time, the $\mathcal{S}$-matrix is not well-defined. Particles in the AdS space-time behave like particles in a box as the null rays reflect back from the time-like boundary. Hence, in the context of AdS, the definition of on-shell asymptotic states is ambiguous, and we do not have any notion of an on-shell $\mathcal{S}$-matrix. A complete theory of quantum gravity in asymptotically AdS spacetime can be understood in terms of a Conformal Field Theory (CFT) sitting at its boundary. This holographic duality, known in the literature as AdS/CFT correspondence, was proposed by Maldacena in the year 1997 [166] and has been greatly applied and explored by others since then [171, 222–231].

What does flat space limit of AdS spacetime signify?

There have been works on scattering amplitudes in AdS spacetime, which helps in understanding the Infrared (IR) finite $\mathcal{S}$-matrix in the flat space limit of AdS/CFT [171, 222, 225–227, 232]. This has various applications, including the $\mathcal{S}$-Matrix bootstrap program [40–42, 230, 231]. In AdS, particles correspond to irreducible representations of the conformal group, allowing for a connection between QFT in flat space and QFT in AdS. This connection is established through the relationship that the $\mathcal{S}$-matrices of flat space can be derived by taking the dimension of the conformal field Δ in the CFT correlator to be large when the dual AdS length scale, denoted by $R \to \infty$. This transition enables a comprehensive understanding of QFT in AdS by utilizing insights from CFT correlation functions and their connection to the flat space formulation[28].

In literature, various works have suggested ways to compute the flat space S-matrix from the position space conformal correlators [223, 224, 234]. In [223], authors showed that, for a massive scalar field, in the flat space limit ($R \to \infty$), the bulk-to-boundary propagator reduces to the usual external leg factors for position space Feynman diagrams, and the bulk-to-bulk propagator indeed reduced to Feynman propagators. Consequently, the correlation functions

[28] Hence AdS radius R serves as the infrared regulator for the flat spacetime [233].

or Witten diagrams [228] also reduce to flat space correlation functions in the large R limit. Interestingly, their approach is valid only in a small region of AdS space where the distances are very small as compared to AdS radius R. This is a direct consequence of the large Δ limit manifesting itself as the large R limit.

A way to construct the on-shell momentum states in bulk is to use the HKLL prescription [223, 224, 234, 235], and the momentum space prescription given by [236] agrees to this analysis at leading order in large R. Another line of work in this direction is done by [50], where the authors obtained the S-matrix of the flat space patch around the center of global AdS from the CFT correlators in a large radius limit. The conformal operators constructed create the asymptotically scattering states of the flat space patch. Here, they have also explored the IR structure of the flat space scattering amplitudes that reproduce Weinberg's soft photon theorems[29].

1.6.1 *Soft* motivations

However, the nature of "soft theorems" in the context of scattering amplitudes in theories in the presence of a cosmological constant remains less explored. While these are better explained within flat spacetimes [12, 43–46, 48, 55–57, 117–121, 237], their characterization becomes less straightforward in the context of AdS. Soft theorems in terms of Ward identities of asymptotic symmetries constrain the physical observables like the scattering amplitudes [11, 12, 15, 47, 188, 238], which has experimental implications both in gauge and gravity theories [17–21].

The definition of soft theorems becomes challenging due to poorly defined asymptotic states within AdS. Moreover, there is an issue while taking the soft limit in AdS[30]. In this context, another interesting limit, namely the *double scaling limit* (DSL), has been introduced by Banerjee et al. [239, 240]. This is defined as a limit when the frequency (ω) of the radiation and the cosmological constant goes to zero (conversely, the AdS radius R approaches infinity), simultaneously keeping their ratio constant. This limit physically uses the fact that when the space-time approaches to flat, these radiations become soft. This limit provides us the small cosmological

[29] In the context of a quantum scattering process within flat spacetime, taking soft limit implies taking the momentum of one of the external particles to zero ($k^\mu \to 0$). Soft theorems are the behavior of the scattering amplitudes under this soft limit [12, 43, 45, 46, 118].

[30] For instance, the relation between mass m, energy ω, and conformal weight Δ for a massive scalar field in AdS is given by, $m^2R^2 = \Delta(\Delta - d)$ and $\omega R = \Delta$. Clearly, $m = 0$ has two distinct solutions, $\Delta = 0$ and $\Delta = d$. The first one is consistent with the definition of soft limit ($m, \omega \to 0$). However, the second solution is inconsistent with $\omega \to 0$ limit [239, 240]. For a finite AdS radius R, any QFT in AdS spacetime is IR regulated [232], which means AdS allows us to probe energy scale only up to $\mathcal{O}(1/R)$. Hence, to establish the soft limit within the AdS framework, enabling us to probe an extremely low-energy regime, we take the AdS radius to be large.

constant corrections to the flat space Classical Soft theorems as described in [239, 240]. Further in [241, 242], it has been shown that the small cosmological constant corrected soft photon theorem can be derived from a large N CFT_3 Ward identity. A perturbed soft photon mode operator on a flat spacetime patch in global AdS_4 has been derived in terms of an integrated expression of the boundary CFT current.

1.7 Unifying Gravity at Infinities

The motivation to unify gravity at infinities comes from Strominger's Infrared (IR) Triangle [11]. So, this can be a single framework relating to the quantum and classical sectors of gravity at the infinities far away from the source(s) that we are aiming for.

- The passage of a gravitational radiation pulse through a nearby detector induces a relative displacement in the detector position. This effect is measurable and is known as *gravitational memory effect*[31][18, 243, 244].

- On the other hand, due to the passage of this gravitational radiation, the difference in the initial and final geometries of the spacetime is related by a BMS supertranslation as explained in section 1.3.1. This means that the Ward identities of this BMS supertranslation symmetry invariance in quantum gravity are expressed as data representing gravitational radiation at the null infinity [15].

- Gravitational scattering amplitudes involving one or multiple soft graviton currents are given by Weinberg's soft graviton theorem, and the BMS supertranslaion Ward identity can be reproduced from this soft theorem [12].

This is how the infrared triangle is formed. However, quite independently, the relation between Memory and soft theorems has been established by Strominger *et. al.* [11, 132]. Memory effect is the Fourier transformation of the soft theorems, which is the Ward identity of the supertranslation symmetry. I have shown this vacuum transition by the action of supertranslation symmetry[32] in section 1.3.1. The radiations coming out of spacetimes cause this transition from one vacuum state to another.

In the above sections, I explained the symmetries and corresponding charges.

[31] The relation between memory and asymptotic symmetries is given by a universal formula which also has been extended for gauge theories [21].

[32] In electrodynamics, we have electromagnetic memory [20, 21] in terms of electromagnetic radiation coming out from the null hypersurfaces.

Hence, one can address this as

Localised charges at null infinities are interpreted as Memory effects!

This relates to two stationary regions of the spacetimes (two equivalent vacua), which is a DC shift of the outgoing metric, which can be sourced from any binary collisions, as shown by Branginsky and Throne [245].

1.7.1 Implications in GW Astronomy

Gravitational waveforms unfold the secrets to symmetries at the null infinities of the boundaries of our asymptotically flat spacetimes. I am focused here on the application of asymptotic symmetries in the sector of gravitational wave observation. I will highlight here some of the recent literature's research highlights for the reader.

...in the Infrared limit, BMS formalism associated with the asymptotic symmetries of flat Mink_4 and e-BMS group is an exact solution of the gravitational field in the far-zone regime. To solve a binary problem, we need to consider the energy and angular momentum fluxes radiated by the binary. In this regard, the *BMS flux balance law* [246–254] is the time evolution of the BMS charges, constrained by Einstein's field equations. The multipole expansion of this BMS flux balance law gives us the radiative multipole moments for radiation-reaction forces of compact objects [254].

In the week field regime, the post-Newtonian (PN) and post-Minkowskian (PM) formalism [255–264] or Effective Field theory approaches [88, 265–272] relates this radiative mode to the observables of the binary sources. However, one needs to change the parameter space to compare and extract the desired results[33]. In consequence of this flux balance law, we have the observable memory effects[34], which are the detectable classical observables [280–283] in advanced ground-based detectors like LIGO, VIRGO, and KAGRA (LVK)-type detectors [284–291] and space-based gravitational wave detectors like LISA [292]. Some other methodologies for computing the classical observables and producing precision gravity results are quantum scattering amplitude techniques [266, 267, 293], and worldline quantum field theortic techniques [294–297]

In these Gravitational wave observations, interpreting the signal in terms of the wave mem-

[33] This needs a coordinate change from Bondi gauge to de Donder gauge, as the desired results of the canonical multipole moments are defined in de Donder gauge [243, 254, 273–276].

[34] Displacement memory, spin memory, and Center-of-mass memory effects [18, 138, 244, 277–279].

ory implies strong evidence of the symmetry constraints in gravity theories. The BMS symmetries help to provide insights into the final stages of binary mergers and properties of the remnants from the merger.

Words from Numerical Relativists

Understanding asymptotic symmetries using numerical relativity techniques is more highlighted these days, considering the advantages coming from Gravitational Wave astronomy for solving the non-linearities of Einstein's field equation in the strong field regime. There have been many recent developments in this direction [298–306]. Numerical relativists extract this waveform from the simulations at a finite distance from the sources and then extrapolate the data up to null infinities to understand the structure of the asymptotics.

It is challenging for relativists to achieve the necessary precession in numerical simulations to study asymptotic symmetries due to the need for accurate large-distance simulations. As a toolkit, numerical relativity deepens our theoretical understanding of the properties of spacetimes and the nature of gravitational phenomena.

In a full non-linear theory, the local flux of radiations cannot be defined at finite distances from the source. Hence, we must consider the linearised perturbations around a *gauge-invariant* blackhole background [307–311], similar to the small deviations from the flat spacetimes. Perturbations around AFSs are similar to fluctuations in the Schwarzschild background at a large distance, and this serves the purpose. This perturbative technique is an effective tool to extract the physics of the gravitational waves generated in a numerically evolved Asymptotically Flat Spacetimes [312–314].

We aim to explore the potential application of flat space holographic techniques in the classical world of gravitational wave astronomy, offering insights into the detection and analysis of these cosmic ripples in our universe.

1.7.2 State-of-the-Arts In Black Hole Physics

Can I apply flat space holographic techniques to Black Holes?

Of course! The recent field of black hole physics has seen significant advancements in the context of holography. Its relation to string theory and quantum gravity has been vastly explored. The idea of the holographic principle to study bulk physics in terms of boundary information helps physicists to understand the black hole information problem, originally formulated by Stephen Hawking [162, 315]. This duality provides a powerful framework for studying the thermodynamics and entropy of black holes. Using techniques from holography, like replica wormholes and island formulas, researchers hope to understand the properties of Hawking radiations, which incorporates the long-standing open problem in black hole thermodynamics [316–318].

Understanding the entanglement entropy in a conformal field theory of a boundary region to the area of a minimal surface in the bulk AdS space using the well-known Ryu-Takayanagi formula has been an instrumental direction of research [319–321]. Some of the recent excitements are in the direction of complexity in the context of black holes. Holography duality proposed that the complexity of quantum states in the boundary theory of the manifold is related to the bulk geometry of the AdS spacetime. This includes complexity$\sim$volume and complexity$\sim$action conjectures [322–325].

String theory and holographic methods provided methods for counting microstates of the black holes, especially in the context of supersymmetric black holes. This relates to the Bekenstein-Hawking entropy and entropy of the black holes in the area of the horizons [326, 327]. Sen developed the quantum entropy function formalism to compute the entropy of extremal black holes in string theory using the AdS/CFT correspondence [328]. These techniques help us to understand the quantum structure of black holes, and as per the suggestions coming from string theory, our information inside the black hole is never lost, though it is encoded in these microstates. Our understanding of recently developed techniques for celestial holography is to make sense of these developments related to microstate counting in the case of supersymmetric black holes.

Towards Blackhole horizon symmetries

It all started with Brown and Henneaux's understanding of the Virasoro symmetries of the near horizon geometry of the extremal black holes, which are to understand the microstates and entropy of at the killing horizons in relation to the states in [329]. There are some other works to relate these symmetries to the states in conformal field theory [329–332]. Recently, Hawking, Perry, and Strominger, in [141, 333], caught the attention of scientists on solving the black hole information paradox using asymptotic symmetries. In the presence of a black hole, we have two asymptotic observers, one sitting at the asymptotics far away from the black hole and another on/near the horizon. This questions the symmetry groups and whether we have the complete BMS group at the horizon or not. Finding the asymptotic isometry generators on the horizon is a real challenge in the presence of *any* blackholes. Soft charge conservation builds correlations of the in/out states at the black hole horizons.

We must understand the BMS symmetries and their connection to black hole geometry. As an asymptotic observer and an observer outside the event horizon of a black hole, one must compute the gravitational memory effects and their properties produced by any incoming shockwave. This memory is called the black hole memory effect [334, 335]. Recent state-of-the-art is to foliate the event horizon in the early and late times, which are related by Chandrasekaran-Flanagan-Prabhu (CFP) supertranslation symmetries [336].

One of the primary motivations for this thesis is to ignite the reader's curiosity about flat space holography and its power to unravel the symmetries of the spacetimes and their quantized modes at the boundary of the spacetimes.

1.8 Plan of Thesis

I will organize the thesis as follows.

- In the preliminary Chapter 2, I will review the basics of the celestial conformal field theory technique that we use in celestial holography to understand the scattering in asymptotically flat spacetimes, as this will be heavily used in the succeeding chapters in the thesis. As an example, I will use the pure Yang-Mills theory and show the celestial map explicitly with the construction of possible OPEs between the symmetry current generators on the celestial operators.

- In Chapter 3, we will use the above CCFT techniques in the theory where we have non-trivial $\mathfrak{u}(N)$ gauge symmetries like in Einstein Yang-Mills theory, where we reduced it to the special case of Einstein-Maxwell's theory in the presence of $\mathfrak{u}(1)$ gauge symmetry. We will address some of the properties of the theory in constructing the symmetry algebra at the infinities of our AFSs.

- In Chapter 4, the motivation is a bit accidental. The initial motivation was to understand and apply the celestial holographic technique in the case of maximally supersymmetric $\mathcal{N} = 8$ supergravity theory (which will be my Chapter 5). However, as we know, we have the mandatory requirement of soft and collinear sectors of the scattering amplitude in any theory for using this CCFT technique. This leads us to this work of finding the soft and collinear limits of every possible interaction vertices of the scattering in supergravity using the amplitudes of the $\mathcal{N} = 4$ super Yang-Mills (SYM) using double copy formalism.

- As explained above, in Chapter 5, I will introduce the celestial superamplitude computation of $\mathcal{N} = 8$ supergravity theory, which is expected to get mapped to the celestial *quasi* on-shell superfield conformal correlator at the boundary sphere at the null infinities. The construction goes the same as the technique explained in Chapter 2 and 3. Our goal here is to look for the possible symmetry extensions at the asymptotic boundary corresponding to the bulk supersymmetry and the $\mathrm{SU}(8)_R$ symmetry. This asymptotic analysis and the extended BMS symmetry current modes (if any) in supergravity underscores our understanding of supersymmetric black holes and the symmetries *on* or *near* the horizon.

- In chapter 6, I will explain part of our recent work in understanding scattering amplitudes in AdS spacetime in the flat space limit (large AdS length). This work is one of the ways to understand the soft particles in the AdS spacetime. The analysis is done, especially in the presence of vector bosons. We used momentum space formalism to compute the propagators in the embedding space, which in the flat space limit was reduced to the Feynman propagators of the flat spacetime. We are motivated to study an effective model to gain a detailed understanding of the formalism used here.

Publications as part of this Thesis:

1. N. Banerjee, T. Rahnuma and R. K. Singh, "*Asymptotic symmetry of four-dimensional Einstein-Yang-Mills and Einstein-Maxwell theory*," JHEP **01** (2022), 033.

2. N. Banerjee, T. Rahnuma and R. K. Singh, "*Soft and collinear limits in $\mathcal{N} = 8$ supergravity using double copy formalism*,"JHEP **04** (2023), 126.

3. N. Banerjee, T. Rahnuma and R. K. Singh, "*Asymptotic symmetry algebra of $\mathcal{N} = 8$ supergravity*," Phys. Rev. D **109** (2024).

4. N. Banerjee, A. Desai, A. Mitra, K. Fernandes, and T. Rahnuma, *work in progress*.

CHAPTER 2

PRELIMINARIES ON CELESTIAL HOLOGRAPHIC TECHNIQUES

2.1 Introduction

The gravitational dynamics in flat spacetime can be understood through holography, a concept pioneered by Juan Maldacena within the context of AdS/CFT correspondence in 1996. The recent uncovering of the relation between the on-shell physics of asymptotically flat theories and 2D CFT has proved to be a powerful tool in the computation of BMS algebra.

Asymptotic symmetry analysis via killing vectors is often tedious and challenging. So far, this prescription has only been used to obtain the asymptotic symmetry groups of pure gravity theory. In three spacetime dimensions, the alternative Chern-Simons formulation of (super)gravity[1] has turned out to be the most useful tool for obtaining the asymptotic symmetric algebra. However, for the four spacetime dimensions that we are currently interested in, gravity does not have a Chern-Simons formulation. Thus, an alternate method for computing the asymptotic symmetry algebra for a theory of gravity in four dimensions is desirable so that the enhanced symmetry group in the presence of supersymmetry and other internal symmetries can be obtained.

Soft and Collinear limits have played an important role in flat space holography [122, 185]. The collinear limit of amplitude turns into an operator product expansion (OPE) of conformal operators of the celestial conformal field theory (CCFT) on the celestial sphere on the boundary [61, 182, 183, 185, 344, 345]. These OPEs can be used to calculate the non-trivial asymptotic symmetries of the theory. The usual method of calculating asymptotic symmetries is by finding conformal Killing vectors and spinors that become intractable in the presence of other fields in the theory. That is where CCFT becomes important. A recent proposal by Taylor et al. asserts that one can calculate the asymptotic symmetries of gravity theories using soft and collinear limits of amplitude in the framework of CCFT. This has been confirmed to give consistent results in the few cases it has been implemented [7, 186, 188]. Hence, the study of soft and collinear limits in gravity theories is important in the context of celestial holography.

[1] asymptotic symmetries for various three dimensional supergravity theories can be found in [139, 337–342] and for higher spin-gravity can be found in [198, 200, 343].

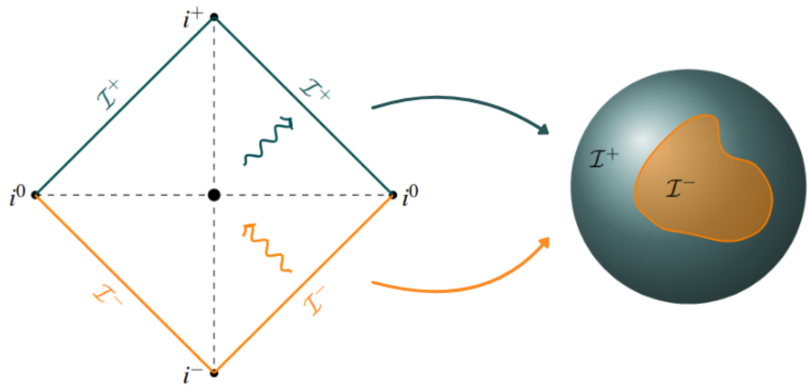


Figure 2.1: Illustration of Celestial Holography. The celestial sphere ($\mathcal{CS}^2$) in RHS is a two-dimensional boundary at the null infinity or $\mathcal{I}^\pm$ of the causal diamond. Points on this sphere correspond to the directions from which light or other massless particles approach or leave the spacetime.

2.2 Celestial Holography

This alternative to flat space holography relates on-shell physics in asymptotically flat theories to a 2D *Celestial* CFT. It maps four-dimensional spacetime symmetries to $\mathcal{CS}^2$ (Fig.2.1). Scattering amplitudes transform into CCFT conformal correlators, yielding BMS generators. Infinite asymptotic symmetries impose constraints on celestial amplitudes through Ward identities, resulting in infinite soft theorems[2]. Recent proposals suggest that these symmetries rely on suitable OPEs, connecting bulk and boundary physics, thus developing the holographic principle in flat spacetime. Let us briefly discuss the relation here:

- It is a well-known fact that the bulk symmetry $\mathrm{SL}(2,\mathbb{C})$ of an asymptotically flat four-dimensional theory is identical to the global part of a two-dimensional Conformal Field Theory (CFT). Given fields in the bulk of an asymptotically flat theory, one can associate conformal operators with these fields that live on the two-dimensional sphere, namely the celestial sphere denoted by $\mathcal{CS}^2$ sitting at the null boundaries of the spacetime.

- The boundary physics is captured by a 2D CFT known as celestial CFT (CCFT) of these operators on $\mathcal{CS}^2$. In this *celestial* approach, the non-trivial symmetries and their algebra of the bulk theory can be computed using the 2D conformal invariance of the CCFT correlators. In particular, the four-dimensional scattering amplitudes of the bulk theory are related via Mellin transformation to the conformal correlators of the CCFT operators.

[2]We have an infinite-dimensional w-symmetry group [346–348]corresonding to the higher spin $2D$ celestial currents, forming an infinite tower of conformally soft graviton/gluon symmetries [349, 350].

Such CCFT correlators that are associated with bulk scattering amplitudes are called *celestial* amplitudes.

2.3 Celestial Map

By now, it is a well-known fact that the scattering amplitudes of a four-dimensional theory in Minkowski spacetime can be cast into appropriate quantities on the celestial sphere via certain mappings of the momenta and amplitudes [179, 181, 344, 351]. The four-dimensional Lorentz group $\mathrm{SL}(2,\mathbb{C})$ is equivalent to the global part of two-dimensional conformal groups and acts on points of the celestial sphere $\mathcal{CS}^2$ via fractional linear transformation. To elaborate on this connection, for four-dimensional spacetime, we use Bondi-Sachs coordinates $(u,r,z,\bar{z})$ where $(z,\bar{z})$ parametrize the celestial sphere at null infinity. Then $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathcal{CS}^2$ as follows:

$$(z,\bar{z}) \longmapsto \left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}).$$

A general null momentum vector $p^\mu : p^2 = 0$ can be parametrized as

$$p^\mu = \omega q^\mu, \quad q^\mu = \frac{1}{2}\left(1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2\right),$$

where q^μ is a null vector, and ω is identified with the light cone energy. Under the Lorentz group, the four-momentum transforms as a Lorentz vector $p^\mu \mapsto \Lambda^\mu{}_\nu p^\nu$. This induces the following transformation of ω and q^μ:

$$\omega \mapsto (cz+d)(\bar{c}\bar{z}+\bar{d})\omega, \quad q^\mu \mapsto q'^\mu = (cz+d)^{-1}(\bar{c}\bar{z}+\bar{d})^{-1}\Lambda^\mu{}_\nu q^\nu.$$

It is useful to introduce the bispinor notation at this stage. We denote the left and right handed helicity spinors by λ_α and $\tilde{\lambda}^{\dot{\alpha}}$ respectively. A given null momentum p^μ can be written as a bispinor

$$p^{\alpha\dot{\alpha}} = \sigma_\mu^{\alpha\dot{\alpha}} p^\mu = \begin{pmatrix} p^0+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0-p^3 \end{pmatrix} = \lambda^\alpha \tilde{\lambda}^{\dot{\alpha}} \tag{2.1}$$

where $\sigma_\mu^{\alpha\dot{\alpha}} = (1, \sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. For real physical momentum, the two spinors are related by complex conjugation $(\tilde{\lambda}^{\dot{\alpha}})^* = \lambda_\alpha$.

We can thus introduce the familiar angle and square bracket spinor notation (see section 1.2.4 for a brief review of spinor-helicity formalism) for the left and right-handed momentum spinors:

$$\lambda^\alpha \equiv \langle p|^\alpha = \sqrt{\omega} \begin{pmatrix} 1 \\ z \end{pmatrix} = \sqrt{\omega}\langle q|^\alpha, \quad \tilde{\lambda}^{\dot{\alpha}} \equiv |p]^{\dot{\alpha}} = \sqrt{\omega} \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} = \sqrt{\omega}|q]^{\dot{\alpha}}, \tag{2.2}$$

where we write

$$\langle q|^\alpha = \begin{pmatrix} 1 \\ z \end{pmatrix}, \quad |q]^{\dot{\alpha}} = \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}. \tag{2.3}$$

To shorten the notation, we denote the spinors for momenta p_i by $\langle i|^\alpha$ and $|i]^{\dot{\alpha}}$ respectively. The inner product of momenta p_i and p_j can then be written in terms of the angle and square brackets of the corresponding spinors, which are now given by

$$\langle ij\rangle = -\sqrt{\omega_i\omega_j} z_{ij}, \quad [ij] = \sqrt{\omega_i\omega_j}\bar{z}_{ij}. \tag{2.4}$$

where $z_{ij} = z_i - z_j$ and similarly $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$. Here, i, j are labels associated with the particles.

Mellin Transformations:

The wave functions in the four-dimensional Minkowski space are mapped to particular operators on the celestial sphere via Mellin transforms. They are also called conformal primary wavefunctions, and the Mellin transformation of the momentum space scattering amplitudes are called *celestial* amplitudes relating the bulk and boundary of our AFSs. These amplitudes are basically the correlation functions in the CCFT.

It's a map between the plane wave basis of the momentum space formulation and the conformal basis. In our case, we have the massless scattering processes, and the wave functions transform as conformal primaries under the Lorentz group. This is done as follows:

For any function $f(\omega)$, the Mellin transform is defined as,

$$\varphi(\Delta) = \int_0^\infty d\omega\ \omega^{\Delta-1} f(\omega). \tag{2.5}$$

Δ could be the “conformal dimension" of the conformal primary wavefunctions corresponding to gauge bosons and graviton. It belongs to the principal continuous series of irreducible unitary $\mathrm{SO}(1,3)$ representation [351], $\Delta \in 1 + i\mathbb{R}$. This is a demand we have to make to define normalizable wave packets. One can further define the inner product of these conformal wavepackets [7]. These conformal dimensions are the Mellin-dual to the energies.

We can now use Mellin integration to transform the fields in bulk to get conformal primaries on $\mathcal{CS}^2$. The spin-0 massless conformal primary of conformal dimension Δ is given by [179, 184, 351]

$$\varphi^{\pm}_{\Delta}(X^\mu, z, \bar{z}) = \int_0^\infty d\omega \omega^{\Delta-1} e^{\pm i\omega q\cdot X - \varepsilon\omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(-q\cdot X \mp i\varepsilon)^\Delta}, \quad \varepsilon > 0. \tag{2.6}$$

The $i\varepsilon$ prescription added here is to make the Mellin integral convergent [184]. Similarly, the plane wave packets corresponding to gauge boson and graviton are given as [185, 186, 352],

$$\varepsilon^\ell_\mu(p) e^{\mp i|p_0|X^0 \pm i\vec{p}\cdot\vec{X}} \quad \text{and} \quad \varepsilon^\ell_{\mu\nu}(p) e^{\mp i|p_0|X^0 \pm i\vec{p}\cdot\vec{X}} \tag{2.7}$$

respectively where ℓ is the helicity and ε^ℓ_μ and $\varepsilon^\ell_{\mu\nu}$ represents the polarisations of the spin-1 and spin-2 particles respectively. Again for spinning particles, we have on $\mathcal{CS}^2$:

$$\begin{aligned} V^{\Delta,\ell}_\mu(X^\mu, z, \bar{z}) &\equiv \partial_J q_\mu \int_0^\infty d\omega\ \omega^{\Delta-1} e^{\mp i\omega q\cdot X - \varepsilon\omega} \quad (\ell = \pm 1) \\ H^{\Delta,\ell}_{\mu\nu}(X^\mu, z, \bar{z}) &\equiv \partial_J q_\mu\ \partial_J q_\nu \int_0^\infty d\omega\ \omega^{\Delta-1} e^{\mp i\omega q\cdot X - \varepsilon\omega}, \quad (\ell = \pm 2) \end{aligned} \tag{2.8}$$

where, $\partial_J = \partial_z$ for $\ell = +1, +2$, $\partial_J = \partial_{\bar{z}}$ for $\ell = -1, -2$. The conformal wave functions corresponding to the graviton and gauge boson, up to gauge and diffeo transformations, respectively, can then be written as:

$$\begin{aligned} A^{\Delta,\ell}_\mu &= g(\Delta) V^{\Delta,\ell}_{\mu J} + \text{gauge} \\ G^{\Delta,\ell}_{\mu\nu} &= f(\Delta) H^{\Delta,\ell}_{\mu\nu} + \text{diffeo} \end{aligned} \tag{2.9}$$

where

$$g(\Delta) = \frac{\Delta - 1}{\Gamma(\Delta+1)}, \quad f(\Delta) = \frac{1}{2}\frac{\Delta(\Delta-1)}{\Gamma(\Delta+2)} \tag{2.10}$$

are the normalization constants[3]. The conformal primaries for nontrivial spins are then given by [7],

$$\begin{aligned}
&\psi^{\pm}_{\Delta,\ell=-1/2;\alpha}(X,z,\bar{z}) = |q\rangle_\alpha \varphi^{\pm}_{\Delta+\frac{1}{2}}(X,z,\bar{z}) \\
&\psi^{\pm;\dot{\alpha}}_{\Delta,\ell=1/2}(X,z,\bar{z}) = |q]^{\dot{\alpha}} \varphi^{\pm}_{\Delta+\frac{1}{2}}(X,z,\bar{z}) \\
&V^{\mu\pm}_{\Delta,\ell=\pm1}(X,z,\bar{z}) = \varepsilon^{\mu}_{\ell=\pm1}(q,r)\varphi^{\pm}_{\Delta}(X,z,\bar{z}) \\
&H^{\mu\nu\pm}_{\Delta,\ell=\pm2}(X,z,\bar{z}) = \varepsilon^{\mu}_{\ell=\pm1}(q,r)V^{\nu\pm}_{\Delta,\ell=\pm1}(X,z,\bar{z}) \\
&\psi^{\mu\pm}_{\Delta,\ell=-3/2}(X,z,\bar{z}) = \varepsilon^{\mu}_{\ell=-1}(q,r)\psi^{\pm}_{\Delta,\ell=-1/2}(X,z,\bar{z}) \\
&\bar{\psi}^{\mu\pm}_{\Delta,\ell=+3/2}(X,z,\bar{z}) = \varepsilon^{\mu}_{\ell=+1}(q,r)\bar{\psi}^{\pm}_{\Delta,\ell=+1/2}(X,z,\bar{z})
\end{aligned} \tag{2.11}$$

where the polarisations are given by

$$\varepsilon^{\mu}_{\ell=+1}(q,r) = \frac{\langle r|\sigma^\mu|q]}{\sqrt{2}\langle rq\rangle}, \quad \varepsilon^{\mu}_{\ell=-1}(q,r) = \frac{[r|\bar{\sigma}^\mu|q\rangle}{\sqrt{2}[qr]} \tag{2.12}$$

with r is a reference null vector and $\bar{\sigma}^\mu \equiv (1, -\sigma_x, -\sigma_y, -\sigma_z)$.

In a scattering process, we take all momenta to be outgoing. We now define *celestial amplitude* or *celestial correlator* on $\mathcal{CS}^2$ as the Mellin transform of the amplitudes:

$$\left\langle \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \equiv \left(\prod_{n=1}^{N} c_n(\Delta_n) \int d\omega_n \omega_n^{\Delta_n-1} \right) \delta^{(4)}\left(\sum_{n=1}^{N} \omega_n q_n \right) A_{\ell_1\ldots\ell_N}(\omega_n, z_n, \bar{z}_n), \tag{2.13}$$

where $A_{\ell_1,\ldots,\ell_n}$ is the bulk amplitude with external particles with helicities $\ell_1,\ldots,\ell_n$, and c_n are the normalization constants.

The celestial correlators can be shown to transform as a conformal correlator under $\mathrm{SL}(2,\mathbb{C})$:

$$\left\langle \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}\left(\frac{az_n+b}{cz_n+d}, \frac{\bar{a}\bar{z}_n+\bar{b}}{\bar{c}\bar{z}_n+\bar{d}} \right) \right\rangle = \prod_{i=1}^{N} (cz_i+d)^{\Delta_i+\ell_i}(\bar{c}\bar{z}_i+\bar{d})^{\Delta_i-\ell_i} \left\langle \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle. \tag{2.14}$$

[3]The presence of normalization constants ($g(\Delta)$, $f(\Delta)$) fixes the fields with spin 1 and spin 2 to be pure gauge and pure diffeomorphisms respectively under soft *conformal* limits [186], $\Delta \to 1$ and $\Delta \to 0, 1$. These factors implement the CPT symmetry of the 4D theory at the level of celestial CFT [185].

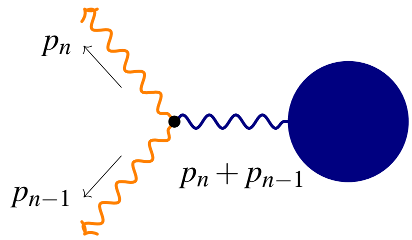


Figure 2.2: Feynman diagram for $p_n||p_{n-1}$, in a $n-$gluon scattering process. The blob represents the rest of the interaction vertices of the scattering process. The collinear singularity occurs when the collinear particles are emitted from the same 3−gluon vertex [122].

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}). \tag{2.15}$$

In the language of celestial CFT, the operator product expansion (OPE) of the conformal operators can be extracted from the correlator in Eq.(2.13) by taking the limits of coinciding insertion points on $\mathcal{CS}^2$. The same implies a collinear limit of momenta in the scattering amplitudes. The form of the OPEs depends on different gauge/gravity conformal operators in various helicity combinations within their collinear limit.

Below, I will discuss the CCFT analysis using the Pure Yang-Mills theory as an example. I will highlight the Soft and Collinear sector of celestial scattering amplitude and how it helps in constructing the asymptotic symmetry algebra. The entire analysis will be shown in the *celestial basis* in the bulk of our Mink$_4$.

2.4 Pure Yang-Mills (YM) theory

Let us discuss the tree-level amplitudes in Yang-Mills theory with two collinear momenta given by the collinear poles [185].

In our context, when two massless particles propagate with parallel four-momenta, they are said to be collinear with each other. The propagator has a pole when a virtual particle splits into a collinear pair via a three-point interaction (Fig.2.2). Such singularities can be separated out via *collinear* limit where the only combined vertex contributes to the divergence. This way, we can factorize our tree amplitudes. Let the combined momentum of the collinear gluons be

$$P^\mu = p^\mu_{n-1} + p^\mu_n = \omega_P q^\mu_P,$$

where the parametrizations are as follows

$$\omega_P = \omega_{n-1} + \omega_n, \quad q_P^\mu = q_{n-1}^\mu = q_n^\mu.$$

The collinear particles emerge from the same 3-gluon vertex. Here, ε_i is the polarization of the ith gluon. Now, the propagator and the vertex contribution in the momentum space are given by [122]

$$\frac{-i}{2p_{n-1}.p_n}(-i)(-g)\Big[(\varepsilon_{n-1}.\varepsilon_n)(p_{n-1}-p_n)^\mu - 2(\varepsilon_n.p_{n-1})\varepsilon_{n-1}^\mu + 2(\varepsilon_{n-1}.p_n)\varepsilon_n^\mu\Big]$$

For instance, in the case of two equal helicity collinear particles ($\varepsilon_{n-1}.\varepsilon_n = 0$),

$$A(1,\cdots n-1^h, n^h) \to \frac{g}{\sqrt{2}}\Big(\frac{\varepsilon_{n-1}.p_n - \varepsilon_n.p_{n-1}}{p_{n-1}.p_n}\Big)A(1,\cdots P^h)$$

In the above, we have considered the color-stripped gluon amplitudes.

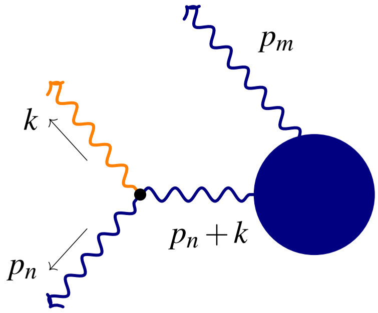
Let's translate this from the momentum basis to celestial basis.

At tree level, the partial amplitudes [4] for possible helicity combinations are given by [185],

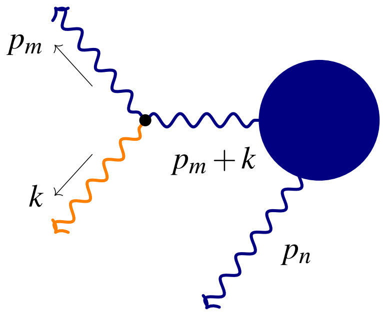
$$\begin{aligned}
\mathcal{M}\left(1,\ldots,n-1^+,n^+\right) &= \frac{1}{z_{(n-1)n}}\frac{\omega_P}{\omega_{n-1}\omega_n}\mathcal{M}\left(1,\ldots,n-2,P^+\right)+\ldots \\
\mathcal{M}\left(1,\ldots,n-1^-,n^-\right) &= \frac{1}{\bar{z}_{(n-1)n}}\frac{\omega_P}{\omega_{n-1}\omega_n}\mathcal{M}\left(1,\ldots,n-2,P^-\right)+\ldots \\
\mathcal{M}\left(1,\ldots,n-1^-,n^+\right) &= \frac{1}{z_{(n-1)n}}\frac{\omega_{n-1}}{\omega_n\omega_P}\mathcal{M}\left(1,\ldots,n-2,P^-\right) \\
&\quad + \frac{1}{\bar{z}_{(n-1)n}}\frac{\omega_n}{\omega_{n-1}\omega_P}\mathcal{M}\left(1,\ldots,n-2,P^+\right)+\ldots.
\end{aligned} \tag{2.16}$$

where $z_{(n-1)n} = z_n - z_{(n-1)}$. Let us explain the notation used in the above expression. In LHS, we have the partial amplitudes $\mathcal{M}\left(1,\ldots,(n-1)^\alpha,n^\beta\right)$ of n gauge bosons with two adjacent gauge bosons $n-1$, n with their specific helicities $\alpha,\beta = \pm 1$. In the RHS, we have $n-1$ point partial gauge boson amplitude with the $(n-1)$-th gauge boson having the combined momenta P of the collinear pair with helicity $\alpha = \pm 1$. The RHS also contains leading collinear poles corresponding to the adjacent bosons. After Mellin transforms these amplitudes as in

[4]Partial amplitudes are the color-stripped amplitudes corresponding to a particular choice of the Chan-Paton factor Tr($T^1T^2\ldots T^n$). See [122] for details.



(a) k gluon attached to nth leg.



(b) k gluon attached to the mth leg.

Figure 2.3: Feynman diagram for the soft ($k \to 0$) singularity, in a $n-$gluon scattering process. This arises due to the pole structure of the intermediate propagator[122], where a virtual gluon goes on-shell by emitting as a soft gluon. The two diagrams are the two possibilities, where the k gluon is either attached to (a) the nth leg or to the (b) mth leg.

Eq.(2.13), we extract all the OPEs corresponding to the partial amplitudes as follows,

$$\begin{aligned}
\mathcal{O}^a_{\lambda_1+}(z,\bar{z})\mathcal{O}^b_{\lambda_2+}(w,\bar{w}) &= \frac{C_1}{z-w}\sum_c f^{abc}\mathcal{O}^c_{(\lambda_1+\lambda_2)+}(w,\bar{w}) + \dots \\
\mathcal{O}^a_{\lambda_1-}(z,\bar{z})\mathcal{O}^b_{\lambda_2-}(w,\bar{w}) &= \frac{C_2}{\bar{z}-\bar{w}}\sum_c f^{abc}\mathcal{O}^c_{(\lambda_1+\lambda_2)-}(w,\bar{w}) \dots \\
\mathcal{O}^a_{\lambda_1-}(z,\bar{z})\mathcal{O}^b_{\lambda_2+}(w,\bar{w}) &= \frac{C_3}{z-w}\sum_c f^{abc}\mathcal{O}^c_{(\lambda_1+\lambda_2)-}(w,\bar{w}) + \frac{C_4}{\bar{z}-\bar{w}}\sum_c f^{abc}\mathcal{O}^c_{(\lambda_1+\lambda_2)+}(w,\bar{w}) + \dots
\end{aligned} \tag{2.17}$$

where $\Delta_i = 1 + i\lambda_i$ and C_i are the normalization constants given in [185].

Likewise, in a scattering process, when a virtual particle goes on-shell, we get a soft particle, which results in soft singularities (Fig.2.3). The color-stripped part of the propagator and vertex contribution is given in momentum space as in [122] as,

For Fig.2.3a in $k \mapsto 0$, the propagator and three-gluon vertex contribute

$$\frac{-i}{2k.p_n}(-i)(-g)\Big[(\varepsilon_k.\varepsilon_n)(k-p_n)^\mu - 2(\varepsilon_n.k)\varepsilon_k^\mu + 2(\varepsilon_k.p_n)\varepsilon_n^\mu\Big]$$

The above result has to be contracted with the ε_n^μ at the end to incorporate Lorentz invariance of the amplitude. One can note here that the first term will vanish as a result of gauge invariance as $(k-p_n)^\mu = -2k.p_n = 0$ under $k \to 0$. Hence, only the last term survives. The net effect is that, up to an overall factor, the (k,n) pair is replaced by a single gluon carrying the momentum and helicity of the nth gluon. Similarly, we have the contribution coming from Fig.2.3b. Hence, we have the color-stripped amplitude as,

$$A(1,2,\cdots m,k,n,\cdots,N) \to \frac{g}{\sqrt{2}}\Big(\frac{\varepsilon_k.p_n}{k.p_n} - \frac{\varepsilon_k.p_m}{k.p_m}\Big)A(1,2,\cdots m,n,\cdots,N).$$

This is up to a relative sign due to the antisymmetricity of the structure constant.

In CCFT, we assign a conformal operator to each of the soft particles whose soft energy $(\omega \to 0)$ limit corresponds to conformal soft limit $\Delta \to 1$ in the case of gauge Bosons and $\Delta \to 0$ and $\Delta \to 1$ in case of gravitons [186]. An $n-$point scattering amplitude in YM theory (in celestial basis) after Mellin transformation is given by,

$$\mathcal{A}_{J_1\ldots J_n}(\Delta_i, z_i, \bar{z}_i) = \left(\prod_{i=1}^{n} g(\Delta_i)\int d\omega_i\ \omega_i^{\Delta_i-1}\right)\delta^{(4)}\left(\sum_i \varepsilon_i\omega_i q_i\right)\mathcal{M}_{\ell_1\ldots\ell_n}(\omega_i, z_i, \bar{z}_i) \qquad (2.18)$$

where the arguments $(\Delta_i, z_i, \bar{z}_i)$ in both sides span over n values.

In YM theory, for any helicity configurations, after taking the soft limit of the nth particle, we have our celestial YM amplitude in Mellin space [185] as,

$$\mathcal{A}_{J_1,J_2,\ldots,J_{n-1}(J_n=+1)} = (-i)\left(\frac{1}{z_{(n-1)n}} + \frac{1}{z_{n1}}\right)\mathcal{A}_{J_1,J_2,\ldots,J_{n-1}}. \qquad (2.19)$$

For negative helicities, we have a similar anti-holomorphic relation.

Using the relation between the correlator and the scattering amplitude defined in Eq.(2.13), one can now write the correlator corresponding to the above amplitude by summing over all Chan-Paton factors [185] (see footnote 4). This gives

$$\left\langle \mathcal{O}^{a_1}_{\Delta_1,J_1}\mathcal{O}^{a_2}_{\Delta_2,J_2}\cdots\mathcal{O}^{a_{n-1}}_{\Delta_{n-1},J_{n-1}}\mathcal{O}^{a_n}_{\Delta_n,J_n}\right\rangle = \sum_{\sigma\in S_{n-1}}\mathcal{A}^{\sigma}_{J_1J_2\ldots J_{n-1}J_n}\,\mathrm{Tr}\left(T^{a_1}T^{a_{\sigma(2)}}\ldots T^{a_{\sigma(n-1)}}T^{a_{\sigma(n)}}\right).$$

This gives us the Ward identity with the soft current, $j^a(z) = \lim_{\Delta \to 1} \mathcal{O}^a_{\Delta,+}(z,\bar{z})$,

$$\begin{aligned} &\left\langle j^a(z)\mathcal{O}^{b_1}_{\Delta_1,J_1}(z_1,\bar{z}_1)\,\mathcal{O}^{b_2}_{\Delta_2,J_2}(z_2,\bar{z}_2)\ldots\mathcal{O}^{b_n}_{\Delta_n,J_n}(z_n,\bar{z}_n)\right\rangle \\ &\quad = \sum_{i=1}^{n}\sum_{c}\frac{f^{ab_ic}}{z-z_i}\left\langle \mathcal{O}^{b_1}_{\Delta_1,J_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,J_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,J_n}(z_n,\bar{z}_n)\right\rangle . \end{aligned} \tag{2.20}$$

Similarly, we can find the Ward identity for the anti-holomorphic current $\bar{j}^a(\bar{z}) = \lim_{\Delta \to 1} \mathcal{O}^a_{\Delta,-}(z,\bar{z})$. These Ward identities can also be derived using OPEs. In the CCFT technique, the soft and collinear limits of the celestial amplitudes in a given theory are the *necessary ingredients* for asymptotic symmetry analysis.

In the next chapter, we have shown the proper asymptotic symmetry analysis with the use of the above CCFT technique in the case of Einstein-Yang-Mills and Einstein-Maxwell Theory.

CHAPTER 3

ASYMPTOTIC SYMMETRY OF EINSTEIN-YANG-MILLS AND EINSTEIN-MAXWELL THEORY IN $4D$

This chapter is based on the work conducted in collaboration with Nabamita Banerjee and Ranveer K. Singh, JHEP 01 (2022) 033, published on 10th January 2022.

3.1 Introduction

In this chapter, we have computed the asymptotic BMS algebra in Einstein-Yang-Mills (EYM) and Einstein-Maxwell (EM) theories using the above-discussed CCFT technique. Both the symmetry algebras are already known in the literature [153, 353], and our results match with them. This analysis checks the alternate prescription of finding asymptotic symmetry algebras for four-dimensional flat theories.

Celestial amplitudes are of immense importance as an independent entity in itself [344]. As stated earlier, there are an infinite number of symmetries in asymptotically flat theories. These symmetries impose constraints on the celestial amplitudes via Ward identities, which give us the soft theorems in terms of celestial amplitudes [185]. Since the bulk scattering amplitudes are related to celestial amplitudes via Mellin transformation, these constraints, in turn, imply constraints on the bulk scattering amplitude. Thus, a more thorough understanding of celestial amplitudes can uncover new symmetries and constraints on the bulk amplitudes.

Given the defined *celestial* amplitude, we then construct conserved currents of the CCFT via shadow transform [351] of the conformal operators and compute the OPEs of various currents. The OPEs give us the algebra of the Laurent modes of the currents using standard methods of 2D CFT. After quantization, these modes act as the generators of the BMS algebra. Thus, the computations of asymptotic symmetry algebra reduce to the computation of appropriate OPEs, which in turn depends on the soft and collinear limits in the bulk.

In Section 2.3, we record the preliminaries for writing the currents, including the properties of conformal primaries, and set up notations for the rest of the paper. In Section 3.2, we define the current corresponding to the conformal gauge Boson operator in EYM theory and write the OPE with spin 1 primary operators. In Section 3.2.2, we derive the OPEs of different combinations of the EYM currents with the supertranslation and superrotation generators of pure

BMS symmetry. We further construct a composite current and elaborate on its significance. In Section 3.3, we compute the symmetry algebra from the modes of currents and find the $\mathrm{u}(N)$ extended $\mathfrak{bms}_4$ algebra. In Section 3.4, we define the current corresponding to the conformal operator in a $\mathrm{u}(1)$ gauge theory and use the results of previous sections with appropriate changes to compute the asymptotic symmetry algebra of EM theory. We end this chapter with discussions and open questions. Finally, in Appendix A, we compute certain integrals using the global conformal invariance of the celestial correlators and verify the conformal dimension of our normalized current using the traditional method of constructing OPE with the superrotation generators.

3.2 Asymptotic Symmetry Generators in EYM Theory

In this section, we look for operators in the corresponding CCFT of EYM theory that will generate the asymptotic symmetry of the theory at null infinity. As per our expectation in this case the asymptotic symmetry algebra will be an extension of the usual $\mathfrak{bms}_4$ algebra by an $\mathrm{u}(N)$ current.

3.2.1 The symmetry currents

We first construct the currents in our theory. Following [186], we define the energy-momentum tensor $T(z)$ (and $\overline{T}(\bar{z})$) as the shadow transform of the $\Delta = 0$ graviton conformal operator $\mathcal{O}_{0,-2}$ (and $\mathcal{O}_{0,+2}$):

$$\begin{aligned} T(z) &= \frac{3!}{2\pi} \int d^2z' \frac{1}{(z-z')^4} \mathcal{O}_{0,-2}\left(z',\bar{z}'\right) \\ \overline{T}(\bar{z}) &= \frac{3!}{2\pi} \int d^2z' \frac{1}{(\bar{z}-\bar{z}')^4} \mathcal{O}_{0,+2}\left(z',\bar{z}'\right) \end{aligned} \tag{3.1}$$

The usual superrotations of $\mathfrak{bms}_4$ are generated by T and $\overline{T}$. The supertranslations are generated by the supertranslation current $\mathcal{P}(z,\bar{z})$ that is constructed from graviton conformal operators with $\Delta = 1$ as in [186]. For this purpose we first look for the currents $P(z)$, $\overline{P}(z)$ which are defined as the level one descendant of the $\Delta = 1$ graviton conformal operator as,

$$\begin{aligned} P(z) &= \partial_{\bar{z}} \mathcal{O}_{1,+2}(z,\bar{z}) \\ \overline{P}(\bar{z}) &= \partial_{z} \mathcal{O}_{1,-2}(z,\bar{z}). \end{aligned} \tag{3.2}$$

Now, the holomorphic and antiholomorphic currents can be written as [186],

$$P(z) = \frac{1}{8\pi i}\oint d\bar{z}\,\mathcal{P}(z,\bar{z}), \qquad \overline{P}(\bar{z}) = \frac{1}{8\pi i}\oint dz\,\mathcal{P}(z,\bar{z}). \tag{3.3}$$

We now define a current corresponding to the conformal operator of the gluon ${A^a}_\mu$ which would generate the u(N)-gauge transformations as

$$\begin{aligned} G^a(z) &= \frac{1}{2\pi}\int d^2z' \frac{1}{(z-z')^2}\mathcal{O}^a_{1,-1}(z',\bar{z}') \\ \overline{G}^a(\bar{z}) &= \frac{1}{2\pi}\int d^2z' \frac{1}{(\bar{z}-\bar{z}')^2}\mathcal{O}^a_{1,+1}(z',\bar{z}'). \end{aligned} \tag{3.4}$$

The OPEs of $T, \overline{T}$ and $P, \overline{P}$ with a primary operator $\mathcal{O}^a_{\Delta,\ell}$ with conformal weights $(h,\bar{h})$ are already calculated in [186]. We record the result here for later use,

$$\begin{aligned} T(z)\mathcal{O}_{\Delta,\ell}(w,\bar{w}) &= \frac{h}{(z-w)^2}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \frac{1}{z-w}\partial_w\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \text{ regular.} \\ \overline{T}(\bar{z})\mathcal{O}_{\Delta,\ell}(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \text{regular.} \end{aligned} \tag{3.5}$$

For $P(z)$ we have

$$\begin{aligned} P(z)\mathcal{O}_{\Delta,\ell}(w,\bar{w}) &= \frac{(\Delta-1)(\Delta+1)}{4\Delta}\frac{1}{z-w}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{ regular} \quad (\ell=\pm 1), \\ P(z)\mathcal{O}_{\Delta,\ell}(w,\bar{w}) &= \frac{(\Delta-1)(\Delta+2)}{4(\Delta+1)}\frac{1}{z-w}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{ regular} \quad (\ell=\pm 2), \end{aligned} \tag{3.6}$$

and similar OPEs hold for $\overline{P}(\bar{z})$ with conjugated poles.

We now calculate the OPE of G^a and $\overline{G}^a$ with spin 1 primary operators $\mathcal{O}^b_{\Delta,\ell}$. For N such primary operators, we have

$$\left\langle G^a(z)\prod_{n=1}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \frac{1}{2\pi}\int d^2z_0 \frac{1}{(z-z_0)^2}\left\langle \mathcal{O}^a_{1,-1}(z_0,\bar{z}_0)\prod_{n=1}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$

Further, using the soft limit as in [185, Eq. (3.32)] we have

$$\begin{aligned}&\left\langle \mathcal{O}^{a}_{1,-1}(z_0,\bar{z}_0)\prod_{n=1}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &\qquad = \sum_{i=1}^{M}\sum_{c}\frac{f^{ab_ic}}{\bar{z}_0-\bar{z}_i}\left\langle \mathcal{O}^{b_1}_{\Delta_1,\ell_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle\end{aligned} \tag{3.7}$$

The above relation implies that

$$\begin{aligned}\left\langle G^a(z)\prod_{n=1}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle &= \sum_{i=1}^{M}\sum_{c}f^{ab_ic}\frac{1}{2\pi}\int d^2z_0\frac{1}{(z-z_0)^2}\frac{1}{\bar{z}_0-\bar{z}_i} \\ &\quad\times\left\langle \mathcal{O}^{b_1}_{\Delta_1,\ell_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle.\end{aligned}$$

To solve the integral we use (cf. [186, Eq. (3.7)])

$$\partial_{z_0}\left(\frac{1}{z-z_0}\right)=\frac{1}{(z-z_0)^2},\quad \partial_{\bar{z}_0}\left(\frac{1}{\bar{z}_0-\bar{z}}\right)=2\pi\delta^{(2)}(z_0-z)\,. \tag{3.8}$$

Thus, the integral after integrating by parts, becomes

$$\int d^2z_0\frac{1}{(z-z_0)^2}\frac{1}{\bar{z}_0-\bar{z}_i}=-\int d^2z_0\left(\frac{1}{z-z_0}\right)\partial_{z_0}\left(\frac{1}{\bar{z}_0-\bar{z}_i}\right)=-\frac{2\pi}{z-z_i}. \tag{3.9}$$

Thus we get the OPE as

$$\begin{aligned}&\left\langle G^a(z)\prod_{n=1}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &\qquad = -\sum_{i=1}^{M}\sum_{c}\frac{f^{ab_ic}}{z-z_i}\left\langle \mathcal{O}^{b_1}_{\Delta_1,\ell_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle\end{aligned} \tag{3.10}$$

This gives the OPE

$$G^a(z)\mathcal{O}^{b}_{\Delta,\ell}(w,\bar{w})=\frac{1}{w-z}\sum_{c}f^{abc}\mathcal{O}^{c}_{\Delta,\ell}(w,\bar{w}). \tag{3.11}$$

Similarly, we have

$$\left\langle \overline{G}^a(\bar{z}) \prod_{n=1}^{M} \mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle$$
$$= -\sum_{i=1}^{M}\sum_c \frac{f^{ab_ic}}{\bar{z}-\bar{z}_i} \left\langle \mathcal{O}^{b_1}_{\Delta_1,\ell_1}(z_1,\bar{z}_1)\dots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\dots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle \tag{3.12}$$

which gives the OPE

$$\overline{G}^a(\bar{z})\mathcal{O}^b_{\Delta,\ell}(w,\bar{w}) = \frac{1}{\bar{w}-\bar{z}}\sum_c f^{abc}\mathcal{O}^c_{\Delta,\ell}(w,\bar{w}). \tag{3.13}$$

3.2.2 The OPEs of symmetry current generators

We now have all the tools required to compute the OPEs of the symmetry currents. The OPEs of combinations of $T,\overline{T}$ and $P,\overline{P}$ have already been computed in [186]. We compute the remaining combinations below.

Constructiof of OPEs: G^aG^b and $\overline{G}^a\overline{G}^b$

We have

$$\left\langle G^a(z)G^b(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \frac{1}{4\pi^2}\int d^2z_0\frac{1}{(z-z_0)^2}\int d^2z_1\frac{1}{(w-z_1)^2}$$
$$\times\left\langle \mathcal{O}^a_{1,-1}(z_0,\bar{z}_0)\mathcal{O}^b_{1,-1}(z_1,\bar{z}_1)\prod_{n=1}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$

Since this OPE involves double soft limit with identical helicity, it does not depend on the order in which take the soft limit. We first take the soft limit corresponding to the gauge index a. We have

$$\left\langle \mathcal{O}^a_{1,-1}(z_0,\bar{z}_0)\mathcal{O}^b_{1,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$= \sum_{i=1}^{M}\sum_c \frac{f^{ab_ic}}{\bar{z}_0-\bar{z}_i}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\dots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\dots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle.$$

We now take the first shadow transform. Using Eq. (3.9), we get

$$\left\langle G^a(z)G^b(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = -\frac{1}{2\pi}\sum_{i=1}^{M}\sum_{c} f^{ab_ic}\int d^2z_1\frac{1}{(w-z_1)^2}\frac{1}{z-z_i} \times\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle.$$

The integral on the right hand side is regular as $z\to w$ as a result of the global conformal invariance of the correlator in the integrand, see Appendix A.1.1.

This implies that the concerned OPE is regular. Hence,

$$G^a(z)G^b(w)\sim \text{ regular.} \tag{3.14}$$

Similarly,

$$\overline{G}^a(\bar{z})\overline{G}^b(\bar{w})\sim \text{regular.} \tag{3.15}$$

$G^a\overline{G}^b, \overline{G}^a G^b$ and the Composite Current

We consider

$$\left\langle G^a(z)\overline{G}^b(\bar{w})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \frac{1}{4\pi^2}\int d^2z_0\frac{1}{(z-z_0)^2}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^2} \times\left\langle \mathcal{O}^{a}_{1,-1}(z_0,\bar{z}_0)\mathcal{O}^{b}_{1,+1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$

This is a double soft limit with opposite helicities and hence we expect that the result to depend on the order in which we take the soft limits. Let us first take the soft limit corresponding to the gauge index a. Using the soft limit,

$$\begin{aligned}&\left\langle \mathcal{O}^{a}_{1,-1}(z_0,\bar{z}_0)\mathcal{O}^{b}_{1,+1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\ &=\sum_c\frac{f^{abc}}{\bar{z}_0-\bar{z}_1}\left\langle \mathcal{O}^{c}_{1,+1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle\\ &+\sum_{i=2}^{M}\sum_c\frac{f^{ab_ic}}{\bar{z}_0-\bar{z}_i}\left\langle \mathcal{O}^{b_1=b}_{1,+1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle.\\ &=\sum_{i=1}^{M}\sum_c\frac{f^{ab_ic}}{\bar{z}_0-\bar{z}_i}\left\langle \mathcal{O}^{b_1=b}_{1,+1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle.\end{aligned} \tag{3.16}$$

and following the calculations of previous section, we have

$$\left\langle G^a(z)\overline{G}^b(\bar{w})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = -\frac{1}{2\pi}\sum_{i=1}^{M}\sum_{c} f^{ab_ic}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^2}\frac{1}{z-z_i} \times\left\langle \mathcal{O}^{b_1=b}_{1,+1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle.$$

We now take the second soft limit. We finally get

$$\left\langle G^a(z)\overline{G}^b(\bar{w})\prod_{n=2}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = -\frac{1}{2\pi}\sum_{i=1}^{M}\sum_{j=2}^{M}\sum_{c,d} f^{ab_ic}f^{b_{i1}b_{ij}d}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^2}\frac{1}{z-z_i}\frac{1}{z_1-z_j} \times\left\langle \mathcal{O}^{b_{i2}}_{\Delta_2,\ell_2}(z_2,\bar{z}_2)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\ldots\mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\ldots\mathcal{O}^{b_{iM}}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle,$$

where for $1\le i\le M$, $2\le j\le M$, we have

$$b_{ij}=\begin{cases} b_j & i\neq j \\ c & i=j, \end{cases} \qquad b_1=b.$$

Let us briefly explain various terms in the above expressions. Splitting the sum over i, the $i=1$ term comes from the first term of the RHS of (3.16) and here $b_{11}=c$. The $i\ge 2$ terms come from the second term of (3.16). Further, the integral is easily seen to be,

$$\begin{aligned}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^2}\frac{1}{z-z_i}\frac{1}{z_1-z_j} &= -\frac{1}{z-z_i}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)}\partial_{\bar{z}_1}\left(\frac{1}{z_1-z_j}\right) \\ &= -\frac{2\pi}{z-z_i}\frac{1}{\bar{w}-\bar{z}_j}, \qquad (i\neq 1).\end{aligned} \tag{3.17}$$

When $i=1$, we have

$$\begin{aligned}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^2}\frac{1}{z-z_1}\frac{1}{z_1-z_j} &= -\frac{2\pi}{z-z_j}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)}\partial_{\bar{z}_1}\left(\frac{1}{z-z_1}+\frac{1}{z_1-z_j}\right) \\ &= -\frac{2\pi}{z-z_j}\left(\frac{1}{\bar{w}-\bar{z}}+\frac{1}{\bar{w}-\bar{z}_j}\right)\end{aligned} \tag{3.18}$$

Substituting these integrals, we get

$$\begin{aligned}
&\left\langle G^a(z)\overline{G}^b(\bar{w})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&\quad=\left[\frac{1}{\bar{w}-\bar{z}}\sum_{j=2}^{M}\sum_{c,d}f^{abc}f^{cb_jd}\frac{1}{z-z_j}+\sum_{j=2}^{M}\sum_{c,d}f^{abc}f^{cb_jd}\frac{1}{z-z_j}\frac{1}{\bar{w}-\bar{z}_j}\right] \\
&\qquad\qquad\times\left\langle\mathcal{O}^{b_2}_{\Delta_2,\ell_2}(z_2,\bar{z}_2)\ldots\mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle \\
&\quad+\sum_{i=2}^{M}\sum_{j=2}^{M}\sum_{c,d}f^{ab_ic}f^{bb_{ij}d}\frac{1}{z-z_i}\frac{1}{\bar{w}-\bar{z}_j}\left\langle\mathcal{O}^{b_{i2}}_{\Delta_2,\ell_2}(z_2,\bar{z}_2)\ldots\right\rangle.
\end{aligned} \tag{3.19}$$

Next we compute $\overline{G}^b(\bar{z})G^a(w)$. Proceeding as in the previous calculation we get

$$\begin{aligned}
&\left\langle \overline{G}^b(\bar{z})G^a(w)\prod_{n=2}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = -\frac{1}{2\pi}\sum_{i=1}^{M}\sum_{c}f^{bb_ic}\int d^2z_1\frac{1}{(w-z_1)^2}\frac{1}{\bar{z}-\bar{z}_i} \\
&\qquad\qquad\times\left\langle\mathcal{O}^{b_1=a}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle \\
&\quad=\left[\frac{1}{w-z}\sum_{j=2}^{M}\sum_{c,d}f^{bac}f^{cb_jd}\frac{1}{\bar{z}-\bar{z}_j}+\sum_{j=2}^{M}\sum_{c,d}f^{bac}f^{cb_jd}\frac{1}{\bar{z}-\bar{z}_j}\frac{1}{w-z_j}\right] \\
&\qquad\qquad\times\left\langle\mathcal{O}^{b_2}_{\Delta_2,\ell_2}(z_2,\bar{z}_2)\ldots\mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\ldots\mathcal{O}^{b_M}_{\Delta_M,\ell_M}(z_M,\bar{z}_M)\right\rangle \\
&\quad+\sum_{i=2}^{M}\sum_{j=2}^{M}\sum_{c,d}f^{bb_ic}f^{ab_{ij}d}\frac{1}{\bar{z}-\bar{z}_i}\frac{1}{w-z_j}\left\langle\mathcal{O}^{b_{i2}}_{\Delta_2,\ell_2}(z_2,\bar{z}_2)\ldots\right\rangle.
\end{aligned}$$

As expected, these OPEs depend on the order in which we take the soft limits. To construct a quantity which is independent of this choice, we consider the following combination,

$$\mathcal{G}^{ab}(z,\bar{z};w,\bar{w}):=G^a(z)\overline{G}^b(\bar{w})-\overline{G}^b(\bar{w})G^a(z)\equiv[G^a(z),\overline{G}^b(\bar{w})].$$

Let us now compute the OPE.

$$
\begin{aligned}
&\left\langle \mathcal{G}^{ab}(z,\bar{z};w,\bar{w}) \prod_{n=2}^{M} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\
&\quad = \sum_{j=2}^{M}\sum_{c,d} f^{abc} f^{cb_jd} \left(\frac{1}{\bar{w}-\bar{z}} \frac{1}{z-z_j} + \frac{1}{z-w} \frac{1}{\bar{w}-\bar{z}_j} \right) \left\langle \dots \mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \\
&\quad + \sum_{j=2}^{M}\sum_{c,d} f^{abc} f^{cb_jd} \left(\frac{1}{z-z_j} \frac{1}{\bar{w}-\bar{z}_j} + \frac{1}{\bar{w}-\bar{z}_j} \frac{1}{z-z_j} \right) \left\langle \dots \mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \\
&\quad + \sum_{i=2}^{M}\sum_{j=2}^{M}\sum_{c,d} \left(f^{ab_ic} f^{bb_ijd} \frac{1}{z-z_i} \frac{1}{\bar{w}-\bar{z}_j} - f^{bb_ic} f^{ab_ijd} \frac{1}{\bar{w}-\bar{z}_i} \frac{1}{z-z_j} \right) \\
&\qquad\qquad \times \left\langle \dots \mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i) \dots \mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle ,
\end{aligned}
$$

where we used the fact that $f^{abc} = -f^{bac}$.

Recollecting terms after a bit of simplification, we have

$$
\begin{aligned}
&\left\langle \mathcal{G}^{ab}(z,\bar{z};w,\bar{w}) \prod_{n=2}^{M} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\
&\quad = \sum_{j=2}^{M}\sum_{c,d} f^{abc} f^{cb_jd} \left(\frac{1}{\bar{w}-\bar{z}} \frac{1}{z-z_j} + \frac{1}{z-w} \frac{1}{\bar{w}-\bar{z}_j} \right) \left\langle \dots \mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \\
&\quad + \sum_{j=2}^{M}\sum_{c,d} \left[\frac{(f^{abc} f^{cb_jd} + f^{ab_jc} f^{bcd})}{(z-z_j)(\bar{w}-\bar{z}_j)} + \frac{(f^{abc} f^{cb_jd} - f^{bb_jc} f^{acd})}{(\bar{w}-\bar{z}_j)(z-z_j)} \right] \left\langle \dots \mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \\
&\quad + \sum_{\substack{i,j=2 \\ i\neq j}}^{M}\sum_{c,d} \left(f^{ab_ic} f^{bb_jd} \frac{1}{z-z_i} \frac{1}{\bar{w}-\bar{z}_j} - f^{bb_ic} f^{ab_jd} \frac{1}{\bar{w}-\bar{z}_i} \frac{1}{z-z_j} \right) \left\langle \mathcal{O}^{b_{i2}}_{\Delta_2,\ell_2}(z_2,\bar{z}_2) \dots \right\rangle
\end{aligned}
$$

The third term on the r.h.s of the above equation exactly vanishes on swapping dummy summation variables $c \leftrightarrow d$ and $i \leftrightarrow j$.

Now using the Jacobi identity for structure constants,

$$
f^{abc} f^{ced} + f^{aec} f^{bcd} + f^{bec} f^{cad} = 0
$$

we get ,

$$
\sum_c (f^{abc} f^{cb_jd} + f^{ab_jc} f^{bcd}) = \sum_c (f^{abc} f^{cb_jd} + f^{b_jac} f^{cbd}) = -\sum_c f^{bb_jc} f^{cad}.
$$

Similarly,

$$\sum_c (f^{abc}f^{cb_jd} - f^{bb_jc}f^{acd}) = -\sum_c f^{b_jac}f^{cbd}. \tag{3.20}$$

Substituting these we get,

$$\begin{aligned}
&\left\langle \mathcal{G}^{ab}(z,\bar{z};w,\bar{w}) \prod_{n=2}^{M} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\
&\quad = \sum_{j=2}^{M} \sum_{c,d} f^{abc}f^{cb_jd} \left(\frac{1}{\bar{w}-\bar{z}} \frac{1}{z-z_j} + \frac{1}{z-w} \frac{1}{\bar{w}-\bar{z}_j} \right) \left\langle \dots \mathcal{O}^d_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \\
&\quad + \sum_{j=2}^{M} \sum_{c,d} \left[\frac{f^{bb_jc}f^{acd}}{(z-z_j)(\bar{w}-\bar{z}_j)} + \frac{f^{b_jac}f^{bcd}}{(\bar{w}-\bar{z}_j)(z-z_j)} \right] \left\langle \dots \mathcal{O}^d_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle
\end{aligned} \tag{3.21}$$

Here the singular terms in the above correlator involve insertions of the operator $\mathcal{O}^c_{1,-1}(z,\bar{z})$ and $\mathcal{O}^c_{1,+1}(z,\bar{z})$ with shadow transformation which are $G^c(z)$ and $\overline{G}^c(\bar{z})$. We have the OPE corresponding to our defined current,

$$\begin{aligned}
&\left\langle \mathcal{G}^{ab}(z,\bar{z};w,\bar{w}) \prod_{n=2}^{M} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\
&\quad = \sum_{j=2}^{M} \sum_{c,d} f^{abc}f^{cb_jd} \left(\frac{1}{\bar{w}-\bar{z}} \frac{1}{z-z_j} + \frac{1}{z-w} \frac{1}{\bar{w}-\bar{z}_j} \right) \left\langle \dots \mathcal{O}^d_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \\
&\quad = \sum_{c,d} \frac{f^{abc}}{\bar{z}-\bar{w}} \left[-\sum_{j=2}^{M} \frac{f^{cb_jd}}{z-z_j} \left\langle \dots \mathcal{O}^d_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \right] \\
&\qquad - \sum_{c,d} \frac{f^{abc}}{z-w} \left[-\sum_{j=2}^{M} \frac{f^{cb_jd}}{\bar{w}-\bar{z}_j} \left\langle \dots \mathcal{O}^d_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle \right]
\end{aligned} \tag{3.22}$$

Using Eq.(3.10) and Eq.(3.12), we can write the bracketed terms in r.h.s as

$$\begin{aligned}
-\sum_{j=2}^{M} \frac{f^{cb_jd}}{z-z_j} \left\langle \dots \mathcal{O}^d_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle &= \left\langle G^c(z) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle, \\
-\sum_{j=2}^{M} \frac{f^{cb_jd}}{\bar{w}-\bar{z}_j} \left\langle \dots \mathcal{O}^d_{\Delta_j,\ell_j}(z_j,\bar{z}_j) \dots \right\rangle &= \left\langle \overline{G}^c(\bar{w}) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\langle \mathcal{G}^{ab}(z,\bar{z};w,\bar{w}) \prod_{n=2}^{M} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle &= \sum_{c,d} \frac{f^{abc}}{\bar{z}-\bar{w}} \left\langle G^c(z) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\
&\quad - \sum_{c,d} \frac{f^{abc}}{z-w} \left\langle \overline{G}^c(\bar{w}) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle
\end{aligned} \tag{3.23}$$

Hence the OPE:

$$G^a(z)\overline{G}^b(\bar{w}) - \overline{G}^b(\bar{w})G^a(z) = \frac{1}{\bar{z}-\bar{w}}\sum_c f^{abc}G^c(z) + \frac{1}{w-z}\sum_c f^{abc}\overline{G}^c(\bar{w}) + \text{regular.}$$

We further extract the finite piece of $\mathcal{G}^{ab}(z,\bar{z};w,\bar{w})$ remaining at $w=z$. To this end, let us define the current

$$\mathcal{G}^{ab}(z,\bar{z}) = \ :\mathcal{G}^{ab}(z,\bar{z};z,\bar{z}): \ = \ :G^a(z)\overline{G}^b(\bar{z}) - \overline{G}^b(\bar{z})G^a(z): \ \equiv \ :[G^a(z),\overline{G}^b(\bar{z})]:. \quad (3.24)$$

Hence the correlator of the normal ordered current is,

$$\begin{aligned}\left\langle \mathcal{G}^{ab}(z,\bar{z})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle &= \sum_{j=2}^{M}\sum_{c,d}\left[\frac{f^{bb_jc}f^{acd}+f^{b_jac}f^{bcd}}{(z-z_j)(\bar{z}-\bar{z}_j)}\right]\left\langle \dots \mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\dots\right\rangle \\ &= \sum_{j=2}^{M}\sum_{c,d}\left[\frac{f^{abc}f^{cb_jd}}{(z-z_j)(\bar{z}-\bar{z}_j)}\right]\left\langle \dots \mathcal{O}^{d}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\dots\right\rangle,\end{aligned} \quad (3.25)$$

where we used Eq. (3.20). In particular, we have the OPE

$$\mathcal{G}^{ab}(z,\bar{z})\mathcal{O}^{c}_{\Delta,\ell}(w,\bar{w}) = \frac{1}{z-w}\frac{1}{\bar{z}-\bar{w}}\sum_{d,e}f^{abd}f^{dce}\mathcal{O}^{e}_{\Delta,\ell}(w,\bar{w}) + \text{regular.} \quad (3.26)$$

OPEs: $TG^a, T\overline{G}^a$ and $\overline{T}G^a, \overline{T}\,\overline{G}^a$

- $T(z)G^a(w)$:

  We have

$$\begin{aligned}&\left\langle T(z)G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &= \lim_{\Delta\to1}\frac{1}{2\pi}\int d^2z_1\frac{1}{(w-z_1)^2}\left\langle T(z)\mathcal{O}^{a}_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &= \lim_{\Delta\to1}\frac{1}{2\pi}\int d^2z_1\frac{1}{(w-z_1)^2}\left[\frac{h}{(z-z_1)^2}\left\langle \mathcal{O}^{a}_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right. \\ &\qquad \left. + \frac{1}{z-z_1}\partial_{z_1}\left\langle \mathcal{O}^{a}_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right] + \text{reg.},\end{aligned}$$

where we used [352, Eq. (3.19)]. Now since $h = 0$, the first term in the square bracket vanishes. To simplify notations, put $W = w - z_1$ and $Z = z - z_1$. We get

$$\left\langle T(z)G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$= \lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\frac{1}{W^2Z}\partial_{z_1}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle + \text{regular}$$
$$= \lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\frac{1}{z-w}\left(\frac{1}{W^2}-\frac{1}{WZ}\right)\partial_{z_1}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle + \text{reg.}$$

Using integration by parts in the first term in the above integral, we get

$$\lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\frac{1}{W^2}\partial_{z_1}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$= -\lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\partial_{z_1}\left(\frac{1}{W^2}\right)\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$= \partial_w\left[\lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\frac{1}{W^2}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right]$$
$$= \left\langle \partial_w G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle,$$

where we used the fact that

$$\partial_{z_1}\frac{1}{W^2} = -\partial_w\frac{1}{W^2}.$$

Next, using integration by parts, we have

$$\lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\frac{1}{WZ}\partial_{z_1}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$= -\lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\left(\frac{1}{W^2Z}+\frac{1}{WZ^2}\right)\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$
$$= \lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\left(\frac{1}{z-w}\frac{1}{WZ}-\frac{1}{WZ^2}\right)\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$
$$-\frac{1}{z-w}\left\langle G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$

where we have used,

$$\frac{1}{W^2Z} = \frac{1}{z-w}\left(\frac{1}{W^2}-\frac{1}{WZ}\right),$$

The first integral is shown to vanish in Appendix A.1.2 as a result of the global conformal invariance of the correlator in the integrand. Putting together everything we get

$$\begin{aligned}&\left\langle T(z)G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\frac{1}{(z-w)^2}\left\langle G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle+\frac{1}{z-w}\left\langle \partial_w G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.\end{aligned} \tag{3.27}$$

This immediately gives the OPE

$$T(z)G^a(w)=\frac{1}{(z-w)^2}G^a(w)+\frac{1}{z-w}\partial_w G^a(w)+\text{regular}. \tag{3.28}$$

- $\overline{T}(\bar{z})G^a(w)$ and $T(z)\overline{G}^a(\bar{w})$:

We have,

$$\begin{aligned}&\left\langle \overline{T}(\bar{z})G^a(w)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\lim_{\Delta\to1}\frac{1}{2\pi}\int d^2z_1\frac{1}{(w-z_1)^2}\left\langle \overline{T}(\bar{z})\mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\lim_{\Delta\to1}\frac{1}{2\pi}\int d^2z_1\frac{1}{(w-z_1)^2}\left[\frac{\bar{h}}{(\bar{z}-\bar{z}_1)^2}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right.\\&\qquad\left.+\frac{1}{\bar{z}-\bar{z}_1}\partial_{\bar{z}_1}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right]+\text{reg.},\end{aligned}$$

where we used [352, Eq. (3.19)]. Using integration by parts in second term and simplifying the notation by putting $W=w-z_1$ and $\bar{Z}=\bar{z}-\bar{z}_1$ we get

$$\begin{aligned}&\left\langle \overline{T}(\bar{z})G^a(w)\prod_{n=2}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\lim_{\Delta\to1}\frac{1}{2\pi}\int d^2z_1\left[\frac{(\bar{h}-1)}{W^2\bar{Z}^2}\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right.\\&\qquad\left.-\frac{1}{\bar{Z}}\left(\partial_{\bar{z}_1}\frac{1}{W^2}\right)\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right]+\text{reg.}\\&=-\lim_{\Delta\to1}\frac{1}{2\pi}\int d^2z_1\,\frac{1}{\bar{Z}}\left(\partial_{\bar{z}_1}\frac{1}{W^2}\right)\left\langle \mathcal{O}^a_{\Delta,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle+\text{reg.}\end{aligned}$$

where we used integration by parts and the fact that $\bar{h} \to 1$ when $\Delta \to 1$.

Now,

$$\partial_{\bar{z}_1} \frac{1}{W^2} = \partial_{\bar{z}} \partial_{z_1} \frac{1}{W} = \partial_{z_1} \partial_{\bar{z}_1} \frac{1}{W} = -2\pi \partial_{z_1} \delta^{(2)}(\bar{z}_1 - \bar{w})$$

So we get,

$$\begin{aligned}
\langle \overline{T}(\bar{z}) G^a(w) \rangle &= \lim_{\Delta \to 1} \bar{h} \int d^2 z_1 \frac{1}{\bar{Z}} \, \partial_{z_1} \delta^{(2)}(\bar{z}_1 - \bar{w}) \left\langle \mathcal{O}^a_{\Delta,-1}(z_1, \bar{z}_1) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n, \ell_n}(z_n, \bar{z}_n) \right\rangle \\
&= - \lim_{\Delta \to 1} \bar{h} \int d^2 z_1 \frac{1}{\bar{Z}} \, \delta^{(2)}(\bar{z}_1 - \bar{w}) \, \partial_{z_1} \left\langle \mathcal{O}^a_{\Delta,-1}(z_1, \bar{z}_1) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n, \ell_n}(z_n, \bar{z}_n) \right\rangle \\
&= - \lim_{\Delta \to 1} \partial_w \left(\frac{\left\langle \mathcal{O}^a_{\Delta,-1}(w, \bar{w}) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n, \ell_n}(z_n, \bar{z}_n) \right\rangle}{\bar{z} - \bar{w}} \right),
\end{aligned}$$

Now the soft limit in the numerator is given by:

$$\begin{aligned}
&\lim_{\Delta \to 1} \left\langle \mathcal{O}^a_{\Delta,-1}(w, \bar{w}) \prod_{n=1}^{M} \mathcal{O}^{b_n}_{\Delta_n, \ell_n}(z_n, \bar{z}_n) \right\rangle \\
&\qquad = \sum_{i=1}^{M} \sum_{c} \frac{f^{ab_i c}}{\bar{w} - \bar{z}_i} \left\langle \mathcal{O}^{b_1}_{\Delta_1, \ell_1}(z_1, \bar{z}_1) \ldots \mathcal{O}^{c}_{\Delta_i, \ell_i}(z_i, \bar{z}_i) \ldots \mathcal{O}^{b_n}_{\Delta_n, \ell_n}(z_n, \bar{z}_n) \right\rangle
\end{aligned} \tag{3.29}$$

Thus we get,

$$\begin{aligned}
\left\langle \overline{T}(\bar{z}) G^a(w) \prod_{n=2}^{M} \mathcal{O}^{b_n}_{\Delta_n, \ell_n}(z_n, \bar{z}_n) \right\rangle &= -\partial_w \left[\sum_{i=2}^{M} \sum_{c} \frac{f^{ab_i c}}{(\bar{w} - \bar{z}_i)(\bar{z} - \bar{w})} \right] \\
&\quad \times \left\langle \mathcal{O}^{b_2}_{\Delta_1, \ell_2}(z_1, \bar{z}_1) \ldots \mathcal{O}^{c}_{\Delta_i, \ell_i}(z_i, \bar{z}_i) \ldots \mathcal{O}^{b_n}_{\Delta_n, \ell_n}(z_n, \bar{z}_n) \right\rangle.
\end{aligned}$$

It is now clear that we will get delta functions $\delta^{(2)}(z-w)$, $\delta^{(2)}(z_i - w)$, and the OPE will be localised at $z = w$, $w = z_i$ respectively and each term will be regular as $z \to w$.

Thus we conclude that,

$$\overline{T}(\bar{z}) G^a(w) \sim \text{ regular.} \tag{3.30}$$

Similarly,

$$T(z) \overline{G}^a(\bar{w}) \sim \text{ regular.} \tag{3.31}$$

- $\overline{T}(\bar{z})\overline{G}^a(\bar{w})$: We have,

$$\begin{aligned}\left\langle \overline{T}(\bar{z})\overline{G}^a(\bar{w})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &= \lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^2}\left\langle \overline{T}(\bar{z})\mathcal{O}^a_{\Delta,+1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &= \lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^2}\left[\frac{\bar{h}}{(\bar{z}-\bar{z}_1)^2}\left\langle \mathcal{O}^a_{\Delta,+1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right. \\ &\quad \left. + \frac{1}{\bar{z}-\bar{z}_1}\partial_{\bar{z}_1}\left\langle \mathcal{O}^a_{\Delta,+1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right] + \text{reg.},\end{aligned}$$

Since $\Delta \to 1$ implies $\bar{h} \to 0$, following previous calculations, we see that

$$\begin{aligned}\left\langle \overline{T}(\bar{z})\overline{G}^a(\bar{w})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle &= \frac{1}{(\bar{z}-\bar{w})^2}\left\langle \overline{G}^a(\bar{w})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &\quad + \frac{1}{\bar{z}-\bar{w}}\left\langle \partial_{\bar{w}}\overline{G}^a(\bar{w})\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.\end{aligned} \tag{3.32}$$

This implies the OPE as

$$\overline{T}(\bar{z})\overline{G}^a(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^2}\overline{G}^a(\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\overline{G}^a(\bar{w}) + \text{regular.} \tag{3.33}$$

$PG^a, P\overline{G}^a$

We have,

$$P(z)G^a(w) = \lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z'\frac{1}{(w-z')^2}P(z)\mathcal{O}^a_{\Delta,-1}(z',\bar{z}')$$

Now by [352], $\mathcal{O}^a_{\Delta,-1}$ is a primary field with conformal weight $h = \frac{\Delta-1}{2}$, $\bar{h} = \frac{\Delta+1}{2}$. Thus using [186, Eq. (3.17)], we have

$$P(z)G^a(w) = \lim_{\Delta\to 1}\frac{1}{2\pi}\int d^2z'\frac{1}{(w-z')^2}\left[\frac{(\Delta-1)(\Delta+1)}{4\Delta}\frac{1}{(z-z')}\mathcal{O}^a_{\Delta+1,-1}(z',\bar{z}') + \text{reg.}\right]$$

Hence,

$$P(z)G^a(w) \sim \text{ regular.} \tag{3.34}$$

Similarly,

$$P(z)\overline{G}^a(\bar{w}) \sim \text{ regular.} \tag{3.35}$$

Similarly, we have

$$\overline{P}(\bar{z})\overline{G}^a(\bar{w}) \sim \text{ regular.} \qquad \overline{P}(\bar{z})\overline{G}^a(\bar{w}) \sim \text{ regular.} \tag{3.36}$$

Thus we have obtained all the required OPEs for the current operators on $\mathcal{CS}^2$.

3.3 Asymptotic Symmetry of EYM Theory

In this section we find the asymptotic symmetry algebra of our theory. As noted earlier, we expect the algebra to be an extension of the usual $\mathfrak{bms}_4$ with $\mathrm{u}(N)$ current. Thus apart from the usual BMS algebra [186] generators $L_n, \bar{L}_m, P_{n-\frac{1}{2},m-\frac{1}{2}}$, we also have $G^a_m, \overline{G}^b_n$, which are the modes of the $\mathrm{u}(N)$ currents. The BMS supertranslation operator $\mathcal{P}(z,\bar{z})$ is a primary operator of dimension $(\frac{3}{2},\frac{3}{2})$ which is mode expanded as,

$$\mathcal{P}(z,\bar{z}) \equiv \sum_{n,m\in\mathbb{Z}} P_{n-\frac{1}{2},m-\frac{1}{2}} z^{-n-1}\bar{z}^{-m-1} \tag{3.37}$$

where $P_{n-\frac{1}{2},m-\frac{1}{2}}$ are the supertranslation generators.

Since, $h=1$ for $G(z)$ and $\bar{h}=1$ for $\overline{G}(\bar{z})$, we have the following mode expansion:

$$G^a(z) = \sum_{m\in\mathbb{Z}} \frac{G^a_m}{z^{m+1}}, \quad \overline{G}^b(\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{\overline{G}^b_n}{\bar{z}^{n+1}} \tag{3.38}$$

where,

$$G^a_m = \frac{1}{2\pi i}\oint dz\, z^m G^a(z), \quad \overline{G}^b_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^n \overline{G}^b(\bar{z}). \tag{3.39}$$

To this end, we use the OPEs that we have computed in the last section to get the algebra of $G_m, \overline{G}_n$. To find the extended algebra we compute the commutators of $G_m, \overline{G}_n$ with usual BMS algebra generators. OPEs of Eq. (3.34), (3.35) and (3.36) immediately implies:

$$\left[G^a_m, P_{n-\frac{1}{2},-\frac{1}{2}}\right] = \left[\overline{G}^a_m, P_{n-\frac{1}{2},-\frac{1}{2}}\right] = \left[G^a_m, P_{-\frac{1}{2},n-\frac{1}{2}}\right] = \left[\overline{G}^a_m, P_{-\frac{1}{2},n-\frac{1}{2}}\right] = 0. \tag{3.40}$$

The remaining supertranslation generators are given by [186],

$$P_{n-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{i\pi(m+1)} \oint d\bar{w}\bar{w}^{m+1} \left[\bar{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}\right] \tag{3.41}$$

Then,

$$\begin{aligned}
\left[G^a_m, P_{n-\frac{1}{2},m-\frac{1}{2}}\right] &= \frac{1}{i\pi(m+1)} \oint d\bar{w}\bar{w}^{m+1} \left[G^a_m, \left[\overline{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}\right]\right] \\
&= -\frac{1}{i\pi(m+1)} \oint d\bar{w}\bar{w}^{m+1} \left(\left[P_{n-\frac{1}{2},-\frac{1}{2}}, \left[G^a_m, \overline{T}(\bar{w})\right]\right] + \left[\overline{T}(\bar{w}), \left[P_{n-\frac{1}{2},-\frac{1}{2}}, G^a_m\right]\right] \right) \\
&= 0
\end{aligned} \tag{3.42}$$

Here we have used the OPE relations of G^a with $\bar{T}$ and the commutation relation in Eq.(3.40). Similarly for the antiholomorphic current, $\left[\overline{G}^a_m, P_{n-\frac{1}{2},m-\frac{1}{2}}\right] = 0$.

Hence, we can write,

$$\left[G^a_m, P_{k,l}\right] = \left[\overline{G}^a_m, P_{k,l}\right] = 0 \tag{3.43}$$

where $m, n \in \mathbb{Z}$ and $k, l \in \mathbb{Z} + \frac{1}{2}$.

The OPE of G^a, $\overline{G}^a$ with $\overline{T}$, T respectively in Eq. (3.30), Eq. (3.31) implies

$$\left[L_m, \overline{G}^a_n\right] = \left[\overline{L}_m, G^a_n\right] = 0. \tag{3.44}$$

OPE in Eq. (3.14) and Eq. (3.15) gives

$$\left[G^a_m, G^b_n\right] = \left[\overline{G}^a_m, \overline{G}^a_n\right] = 0. \tag{3.45}$$

Next, we compute the commutator of G_n^a with Virasoro generators. We have

$$\begin{aligned}
[L_n, G_m^a] &= \frac{1}{(2\pi i)^2} \oint dz \oint dz' z^{n+1} z'^m \left[T(z), G^a(z')\right] \\
&= \frac{1}{(2\pi i)^2} \oint dz \oint dz' z^{n+1} z'^m \left[\frac{1}{(z-z')^2} G^a(z') + \frac{1}{z-z'} \partial_{z'} G^a(z')\right] \\
&= \frac{1}{2\pi i} \oint dz' z'^m (n+1) z'^n G^a(z') - \frac{1}{2\pi i} \oint dz' (m+n+1) z'^{m+n} G^a(z') \\
&= (n+1) G^a_{m+n} - (m+n+1) G^a_{m+n} \\
&= -m G^a_{m+n}
\end{aligned}$$

where we used integration by parts. Thus we have

$$[G_m^a, L_n] = m G^a_{m+n}. \tag{3.46}$$

Similarly,

$$[\overline{G}_m^a, \overline{L}_n] = m \overline{G}^a_{m+n}. \tag{3.47}$$

We also have a composite current $\mathcal{G}^{ab}(z,\bar{z})$ in our theory. We have the following Laurent expansion for $\mathcal{G}^{ab}(z,\bar{z})$ (see Appendix A.2):

$$\mathcal{G}^{ab}(z,\bar{z}) = \sum_{n,m \in \mathbb{Z}} \mathcal{G}^{ab}_{mn} z^{-m-1} \bar{z}^{-n-1}, \tag{3.48}$$

with

$$\mathcal{G}^{ab}_{mn} = \frac{1}{(2\pi i)^2} \oint dz \oint d\bar{z}\, z^m \bar{z}^n \mathcal{G}^{ab}(z,\bar{z}). \tag{3.49}$$

From the OPE of Eq. (3.26), we have

$$\begin{aligned}
\left[\mathcal{G}^{ab}_{mn}, \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right] &= \frac{1}{(2\pi i)^2} \oint dz \oint d\bar{z}\, z^m \bar{z}^n \left[\mathcal{G}^{ab}(z,\bar{z}), \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right] \\
&= \sum_{d,e} f^{abd} f^{dce} \frac{1}{(2\pi i)^2} \oint dz \oint d\bar{z}\, z^m \bar{z}^n \frac{1}{z-w} \frac{1}{\bar{z}-\bar{w}} \mathcal{O}^e_{\Delta,\ell}(w,\bar{w}) \\
&= w^m \bar{w}^n \sum_{d,e} f^{abd} f^{dce} \mathcal{O}^e_{\Delta,\ell}(w,\bar{w}).
\end{aligned}$$

In a similar way, using the OPEs in Eq. (3.11), (3.13) and the integral expression (3.39) for the modes G^a_m and $\overline{G}^b_n$, we have

$$\left[G^a_m, \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right] = -w^m \sum_d f^{acd} \mathcal{O}^d_{\Delta,\ell}(w,\bar{w})$$
$$\left[\overline{G}^a_n, \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right] = -\bar{w}^n \sum_d f^{acd} \mathcal{O}^d_{\Delta,\ell}(w,\bar{w}).$$

Thus we have

$$\begin{aligned}
\left[\left[G^a_m, \overline{G}^b_n\right], \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right] &= \left[G^a_m, \left[\overline{G}^b_n, \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right]\right] - \left[\overline{G}^b_n, \left[G^a_m, \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right]\right] \\
&= -\bar{w}^n \sum_d f^{bcd} \left[G^a_m, \mathcal{O}^d_{\Delta,\ell}(w,\bar{w})\right] + w^m \sum_d f^{acd} \left[\overline{G}^b_n, \mathcal{O}^d_{\Delta,\ell}(w,\bar{w})\right] \\
&= w^m \bar{w}^n \left[\sum_{d,e} f^{bcd} f^{ade} \mathcal{O}^e_{\Delta,\ell}(w,\bar{w}) - \sum_{d,e} f^{acd} f^{bde} \mathcal{O}^e_{\Delta,\ell}(w,\bar{w})\right] \\
&= w^m \bar{w}^n \sum_{d,e} f^{abd} f^{dce} \mathcal{O}^e_{\Delta,\ell}(w,\bar{w}) \\
&= \left[\mathcal{G}^{ab}_{mn}, \mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right].
\end{aligned}$$

The above expression further justifies the equality of the two operators as

$$\mathcal{G}^{ab}_{mn} = [G^a_m, \overline{G}^b_n] = G^a_m \overline{G}^b_n - \overline{G}^b_n G^a_m. \tag{3.50}$$

Next we look for,

$$\begin{aligned}
\left[\mathcal{G}^{ab}_{mn}, L_k\right] &= \left[G^a_m \overline{G}^b_n - \overline{G}^b_n G^a_m, L_k\right] \\
&= G^a_m \left[\overline{G}^b_n, L_k\right] + [G^a_m, L_k] \overline{G}^b_n - \overline{G}^b_n [G^a_m, L_k] - \left[\overline{G}^b_n, L_k\right] G^a_m \\
&= m G^a_{m+k} \overline{G}^b_n - m \overline{G}^b_n G^a_{m+k} \\
&= m \mathcal{G}^{ab}_{m+k,n}.
\end{aligned} \tag{3.51}$$

Similarly,

$$\left[\mathcal{G}^{ab}_{mn}, \bar{L}_k\right] = n \mathcal{G}^{ab}_{m,n+k} \tag{3.52}$$

Finally, to commute the commutator of $\mathcal{G}^{ab}_{mn}$ with $\mathcal{G}^{a'b'}_{kl}$, we compute its commutator with an arbitary primary operator $\mathcal{O}^c_{\Delta,\ell}(w,\bar{w})$. We have

$$\begin{aligned}
\left[\left[\mathcal{G}^{ab}_{mn},\mathcal{G}^{a'b'}_{kl}\right],\mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right] &= \left[\mathcal{G}^{ab}_{mn},\left[\mathcal{G}^{a'b'}_{kl},\mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right]\right]-\left[\mathcal{G}^{a'b'}_{kl},\left[\mathcal{G}^{ab}_{mn},\mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right]\right]\\
&= w^k\bar{w}^l\sum_{d,e} f^{a'b'd}f^{dce}\left[\mathcal{G}^{ab}_{mn},\mathcal{O}^e_{\Delta,\ell}(w,\bar{w})\right]\\
&\qquad - w^m\bar{w}^n\sum_{d,e} f^{abd}f^{dce}\left[\mathcal{G}^{a'b'}_{kl},\mathcal{O}^e_{\Delta,\ell}(w,\bar{w})\right]\\
&= w^{m+k}\bar{w}^{n+l}\left[\sum_{d,e} f^{a'b'd}f^{dce}\sum_{d',e'} f^{abd'}f^{d'ee'}\mathcal{O}^{e'}_{\Delta,\ell}(w,\bar{w})\right.\\
&\qquad\left. -\sum_{d,e} f^{abd}f^{dce}\sum_{d',e'} f^{a'b'd'}f^{d'ee'}\mathcal{O}^{e'}_{\Delta,\ell}(w,\bar{w})\right]\\
&= \sum_{d,d'} f^{abd'}f^{a'b'd}\left(w^{m+k}\bar{w}^{n+l}\sum_{e,e'} f^{d'de}f^{ece'}\mathcal{O}^{e'}_{\Delta,\ell}(w,\bar{w})\right)\\
&= \sum_{d,d'} f^{abd}f^{a'b'd}\left[\mathcal{G}^{d'd}_{m+k,n+l},\mathcal{O}^c_{\Delta,\ell}(w,\bar{w})\right].
\end{aligned}$$

where we have used the identity of structure constants for simplification. Hence we conclude that

$$\left[\mathcal{G}^{ab}_{mn},\mathcal{G}^{a'b'}_{kl}\right]=\sum_{d,d'} f^{abd'}f^{a'b'd}\mathcal{G}^{d'd}_{m+k,n+l}. \tag{3.53}$$

Now collecting all the commutators we get the complete set of algebra as,

$$\begin{aligned}
&[L_m,L_n]=(m-n)L_{m+n} &&\left[\mathcal{G}^{ab}_{mn},P_{k,l}\right]=0\\
&[\bar{L}_m,\bar{L}_n]=(m-n)\bar{L}_{m+n} &&\left[\mathcal{G}^{ab}_{mn},L_k\right]=m\mathcal{G}^{ab}_{m+k,n}\\
&[L_n,P_{kl}]=\left(\frac{1}{2}n-k\right)P_{n+k,l} &&\left[\mathcal{G}^{ab}_{mn},\bar{L}_k\right]=n\mathcal{G}^{ab}_{m,n+k},\\
&[\bar{L}_n,P_{kl}]=\left(\frac{1}{2}n-l\right)P_{k,n+l} &&\left[\mathcal{G}^{ab}_{mn},\mathcal{G}^{a'b'}_{kl}\right]=\sum_{d,d'} f^{abd'}f^{a'b'd}\mathcal{G}^{d'd}_{m+k,n+l}.
\end{aligned} \tag{3.54}$$

The above algebra (3.54) is our $\mathfrak{bms}_4$ with an $\mathrm{u}(N)$ extension [186]. This is the infinite-dimensional symmetry algebra of Einstein-Yang-Mills theory on $\mathcal{CS}^2$ and can be compared to the asymptotic symmetry algebra of the same as given in [353]. On comparing with [353], we see that our method gives exactly the same algebra as theirs, but in a different basis. In particular, we see that the generator $j^{m,n}_i$ of [353] is related to our generator $\mathcal{G}^{ab}_{mn}$ as follows:

$$\mathcal{G}^{ab}_{mn}=\sum_c f^{abc}j^{m,n}_c.$$

Thus we see that the asymptotic symmetry algebra $\mathfrak{eymbms}_4$ of EYM-theory has the structure of the semi-direct sum of superrotations with the direct sum of the abelian algebra of super-translations and nonabelian $\mathrm{u}(N)$-gauge transformations represented by $\mathcal{G}^{ab}_{mn}$:

$$\mathfrak{eymbms}_4 = \text{Superrotations} \uplus [\text{Supertranslations} \oplus \mathrm{u}(N)\text{-gauge transformations}]\,.$$

In the above formalism by introducing the conformal operator for $\mathrm{u}(N)$ gauge bosons, we have generated the symmetry algebra of the corresponding theory. In the next section we follow the same procedure to find the asymptotic symmetry algebra of Einstein-Maxwell theory by introducing the conformal operator of a single gauge boson.

3.4 Asymptotic Symmetry of EM Theory

In this section we look for the asymptotic symmetry for Einstein-Maxwell theory, using the OPEs of the corresponding $\mathcal{CS}^2$ amplitudes. Following the same prescription as of the last section we define the current corresponding to the conformal operator of the photon A^μ which is supposed to generate the $\mathrm{u}(1)$-gauge transformations in a similar way:

$$\begin{aligned} G(z) &= \frac{1}{2\pi}\int d^2z' \frac{1}{(z-z')^2}\mathcal{O}_{1,-1}(z,\bar{z}) \\ \overline{G}(\bar{z}) &= \frac{1}{2\pi}\int d^2z' \frac{1}{(\bar{z}-\bar{z}')^2}\mathcal{O}_{1,+1}(z,\bar{z}) \end{aligned} \tag{3.55}$$

We now calculate the OPE of G and $\overline{G}$ with spin 1 primary operators $\mathcal{O}_{\Delta,\ell}$. For M such primary operators, we have by Eq.(2.13),

$$\left\langle G(z)\prod_{n=1}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \frac{1}{2\pi}\int d^2z_0 \frac{1}{(z-z_0)^2}\left\langle \mathcal{O}_{1,-1}(z_0,\bar{z}_0)\prod_{n=1}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle .$$

When the gauge group is $\mathrm{u}(1)$, the soft theorem does not involve the gauge group factors. In particular for M bosons interacting in EM-theory, the correlator of the corresponding conformal operators $\mathcal{O}_{\Delta_i,\ell_i}$ is related to the scattering amplitude $\mathcal{A}_{\ell_1,\ldots,\ell_M}$ in Mellin space as follows (cf. [185, Eq. 3.30]):

$$\left\langle \prod_{n=1}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \sum_{\sigma\in S_{M-1}} \mathcal{A}^{\sigma}_{\ell_1,\ldots,\ell_M}(z_n,\bar{z}_n), \tag{3.56}$$

where $\mathcal{A}^{\sigma}_{\ell_1,\ldots,\ell_M}$ is the partial amplitude corresponding to the permutation of the $M-1$ external legs fixing the first leg. The soft limit of [185, Eq. (3.32)] transforms to

$$\begin{aligned}\Big\langle \mathcal{O}_{1,-1}(z_0,\bar z_0)\prod_{n=1}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar z_n)\Big\rangle \\ = \sum_{i=1}^{M}\frac{1}{\bar z_0-\bar z_i}\langle \mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar z_1)\ldots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar z_i)\ldots\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar z_n)\rangle\end{aligned} \tag{3.57}$$

Using this and following the calculations on previous sections, we easily see that

$$\begin{aligned}\Big\langle G(z)\prod_{n=1}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar z_n)\Big\rangle &= -\sum_{i=1}^{M}\frac{1}{z-z_i}\langle \mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar z_1)\ldots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar z_i)\ldots\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar z_n)\rangle \\ \Big\langle \overline{G}(\bar z)\prod_{n=1}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar z_n)\Big\rangle &= -\sum_{i=1}^{M}\frac{1}{\bar z-\bar z_i}\langle \mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar z_1)\ldots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar z_i)\ldots\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar z_n)\rangle\end{aligned} \tag{3.58}$$

This gives the OPE

$$\begin{aligned}G(z)\mathcal{O}_{\Delta,\ell}(w,\bar w) &= \frac{1}{w-z}\mathcal{O}_{\Delta,\ell}(w,\bar w) \\ \overline{G}(\bar z)\mathcal{O}_{\Delta,\ell}(w,\bar w) &= \frac{1}{\bar w-\bar z}\mathcal{O}_{\Delta,\ell}(w,\bar w).\end{aligned} \tag{3.59}$$

The OPEs of Sections 3.2.2, 3.2.2 and 3.2.2 remain the same without the gauge indices. Finally, we see that due to the absence of gauge factors f^{abc}, the OPE of the new current

$$\mathcal{G}(z,\bar z) = \ :G(z)\overline{G}(\bar z)-\overline{G}(\bar z)G(z):$$

is regular:

$$\Big\langle \mathcal{G}(z,\bar z)\prod_{n=1}^{M}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar z_n)\Big\rangle \sim \text{regular.} \tag{3.60}$$

To find out the algebra, we Laurent expand the currents:

$$G(z) = \sum_{m\in\mathbb{Z}}\frac{G_m}{z^{m+1}}, \quad \overline{G}(\bar z) = \sum_{n\in\mathbb{Z}}\frac{\overline{G}_n}{\bar z^{n+1}} \tag{3.61}$$

where,

$$G_m = \frac{1}{2\pi i}\oint dz\, z^m G(z), \quad \overline{G}_n = \frac{1}{2\pi i}\oint d\bar z\, \bar z^n\overline{G}(\bar z). \tag{3.62}$$

The algebra remains the same as in Eq. (3.54) except that now the commutator

$$[G_m, \overline{G}_n] = 0,$$

as can easily be verified using the OPE in Eq. (3.59) and following the same method as in the proof of Eq. (3.50). We collect the algebra here for completeness:

$$\begin{aligned}
\left[G_m, P_{k,l}\right] &= \left[\overline{G}_m, P_{k,l}\right] = 0 \\
\left[L_m, \overline{G}_n\right] &= \left[\overline{L}_m, G_n\right] = 0 \\
\left[G_m, G_n\right] &= \left[\overline{G}_m, \overline{G}_n\right] = \left[G_m, \overline{G}_n\right] = 0 \\
\left[G_m, L_n\right] &= mG_{m+n} \\
\left[\overline{G}_m, \overline{L}_n\right] &= m\overline{G}_{m+n}.
\end{aligned} \tag{3.63}$$

From the algebra above and the algebra of supertranslations and superrotations, we conclude that the extended $\mathfrak{bms}_4$ algebra of EM-theory has the structure of semidirect sum of superrotations with the direct sum of the abelian algebras of supertranslations and $\mathrm{u}(1)$-gauge transformations [153]:

$$\mathfrak{embms}_4 = \text{Superrotations} \uplus \left[\text{Supertranslations} \oplus \mathrm{u}(1)\text{-gauge transformations}\right].$$

We remark that the asymptotic algebra calculated using the methods of [153] only gives the global Lorentz transformations although we have recovered the complete local superrotations algebra using our methods.

3.5 Conclusions and Open Problems

In this chapter, we have used the CCFT technique to compute the asymptotic symmetry algebra of EYM and EM theories in four spacetime dimensions. We defined the celestial conformal operators corresponding to the symmetry currents of graviton and gauge boson fields of the bulk theory and computed their OPEs using the well known soft and collinear limits [122]. The resultant correlators, in their singular (OPE) limit gives the bulk scattering amplitudes where one or more of the incoming particles are soft. Using the standard methods of 2D CFT we computed the symmetry algebra of modes of these current operators, that finally gives us the asymptotic symmetry algebra of the bulk theory. The asymptotic symmetry algebra of the EM theory is obtained from that of the EYM theory, by setting the non-abelian structure constants to zero value. These give us the extended $\mathfrak{bms}_4$ symmetry algebra, in presence of non-abelian and abelian spin 1 current. In the case of EYM theory, due to its non-abelian structure, a composite current conformal operator playes an important role. Our results match with the known results in literature [153, 353] upto a generator redefinition. This bolsters the proposal of Taylor *et. al.* [7, 185, 186, 344, 353, 354] of using celestial CFT correlators to compute asymptotic symmetry algebra of flat field theories.

Recently, the CCFT approach has been extended to calculate the asymptotic symmetry algebra of $\mathcal{N} = 1$ supergravity theory [7]. The asymptotic symmetry algebra, in this case, is an infinite dimensional extension of the $\mathfrak{bms}_4$ algebra by the fermionic current modes. This work made use of supersymmetric ward identities to extract the soft and collinear limits of conformal currents associated with gravitons and gravitinos. It would be interesting to extend this approach to other four-dimensional $\mathcal{N} > 1$ supersymmetric field and gravity theories. In $\mathcal{N} > 1$ supersymmetric theories, due to the presence of R-charges, it is expected that the resultant extension of the asymptotic algebra will be "nontrivial". One prime ingredient of the CCFT approach is the soft and collinear limits of the (super)conformal operators corresponding to the particle excitations in the theory. Thus, the extension to $\mathcal{N} > 1$ case requires the study of soft and collinear singularities in those theories. Such a study appeared in [221] for $\mathcal{N} = 4$ Super Yang-Mills theory. It would be interesting to use their results for finding the asymptotic symmetry algebra for $\mathcal{N} = 4$ Super Yang-Mills theory. However, an even more interesting scenario would be to study the extension of the BMS algebra for supergravity theories with $\mathcal{N} > 1$ supersymmetry. One such candidate is the $\mathcal{N} = 8$ supergravity (I will do the complete asymptotic analysis in Chapter 5), which is related to $\mathcal{N} = 4$ Super Yang-Mills theory via the double copy relations, and hence, an analysis of soft and collinear limits would be tractable in this case.

In the next chapter, I will report these using the double copy relations, keeping the motivation for the symmetry construction in $\mathcal{N} = 8$ supergravity in mind.

CHAPTER 4

SOFT AND COLLINEAR LIMITS USING DOUBLE COPY FORMALISM

The work mentioned in this chapter was conducted in collaboration with Nabamita Banerjee and Ranveer K. Singh, JHEP 04 (2023), published on 26th April 2023.

4.1 Introduction

Although the dynamics in gravity and the gauge theory are different, we can establish relations between these two. The one well-known relation is in the context of AdS/CFT correspondence, where we have the relations between the weakly coupled gravity and strongly coupled quantum field theory. Here, we will be specific to the weak coupling regime only, which is better described by the recently developed in-hand technique of Double copy formalism. A brief review of the formalism is given in section 1.5, showing the duality using this multiplicative bilinear operation.

As we know, double copy formalism is well-suited to construct on-shell scattering amplitudes in gravity theories from on-shell amplitudes in gauge theories. However, there have been off-shell developments for double-copy constructions in the context of field theory. The covariant fields in both gauges get double-copied to give rise to gravity fields, which give rise to the map at the level of Lagrangian [355–360]. As an application to the perturbative classical gravity, the off-shell methods are applied to worldline effective field theories [361–369]. Some advancements are in the case of perturbative double copies of the classical fields, which are generalizations for the on-shell KLT relations [101, 370–375]. Considering the exact solutions of the general relativity to be Schwarzschild and Kerr black holes [376], we have Kerr-Schild Double copy [377–379] and Weyl double copy [380, 381] as some of the exact double copy formulations in classical gravity. These results hint towards the geometric formulation of the theory of general relativity beyond the realm of perturbative analysis [382].

This chapter focuses on the on-shell methods of the scattering amplitudes and is explicitly dedicated to the low-energy (IR) sectors, like soft and collinear sectors of the scattering amplitudes in both gauge and gravity theories. Here, we calculate the soft and collinear limits of all possible helicity combinations in $\mathcal{N}=8$ supergravity using the known double copy relation to

$\mathcal{N}=4$ SYM. General formulas for the double copy of amplitudes exist in literature [216, 217], but to our knowledge, they have not been worked out explicitly.

Other than the case reviewed in this thesis, in the case of maximally supersymmetric theories, there are other double copy relations successfully incorporated in different theories, like in the effective theory of the Non-Linear Sigma Models (NLSM) [383–385]. Scattering amplitudes in theories with extended $\mathcal{N}\geq 4$ supersymmetry are given in terms of the double copy relations involving $\mathcal{N}=4$ super Yang-Mills (SYM), Yang-Mills (YM) theory or other SYM theories [386]. For a review of the web of theories at the tree and loop level of the quantum amplitudes, the reader must visit the work by Bern et al. [387].

In Section 4.2, we briefly review soft and collinear limits in $\mathcal{N}=4$ SYM, which we use later in the chapter. Double copy and the relevant formula relating to the amplitudes are reviewed in Section 1.5.1. In Section 4.3, we recall some basic facts about $\mathcal{N}=8$ supergravity and state our conventions for its factorization into a pair of $\mathcal{N}=4$ SYM theories. Finally, in Sections 4.4 and 4.5, we record the explicit soft and collinear limits of supergravity amplitudes. In the main body, we have tabulated the collinear limits of the amplitudes with the appropriate R-symmetry indices, and the detailed calculations have been postponed to the appendices for reference. The appendices also include spinor-helicity formalism and a list of computational results.

4.2 Soft and Collinear Limits in $\mathcal{N}=4$ SYM

As detailed in the introduction, in this chapter, we shall be studying the interesting limits of supergravity amplitudes using double copy relations. For this purpose, we use $\mathcal{N}=4$ SYM as a machinery to find our desired results for gravity. Let us briefly recall some of the prime properties of $\mathcal{N}=4$ SYM. There are 16 different fields in $\mathcal{N}=4$ SYM, all of which can be packaged in a single superfield. Let $\{\eta_a\}_{a=1}^{4}$ be the Grassmann odd coordinates on the superspace. Then the superfield for $\mathcal{N}=4$ SYM can be written as

$$\begin{aligned}\Psi(p,\eta)=&G^{+}(p)+\eta_a\Gamma_{+}^{a}(p)+\frac{1}{2!}\eta_a\eta_b\Phi^{ab}(p)\\&+\frac{1}{3!}\varepsilon^{abcd}\eta_a\eta_b\eta_c\Gamma_{d}^{-}(p)+\frac{1}{4!}\varepsilon^{abcd}\eta_a\eta_b\eta_c\eta_d G^{-}(p)\end{aligned}\tag{4.1}$$

where $G^{\pm}(p)$ denote positive and negative helicity gluons, $\Gamma_{+}^{a},\Gamma_{a}^{-}$ denote positive and negative helicity gluinos respectively and Φ^{ab} denotes the scalars. The superamplitude of n such superfields is then given by the n-point correlation function

$$\mathcal{A}_n(\{p_1,\eta^1\},\ldots\{p_n,\eta^n\}) \equiv \langle \Psi_1(p_1,\eta^1)\ldots\Psi_n(p_n,\eta^n)\rangle. \tag{4.2}$$

We sometimes suppress the momenta p_i and superspace Grassmann coordinates η^i and simply write $\mathcal{A}_n(1,2,\ldots,n)$. Expanding both sides in η and comparing, one gets the scattering amplitude of all the component fields. Next, we find the soft and collinear limits of the superamplitude. We begin with the soft theorem following [113]:

$$\mathcal{A}_n(\cdots,a,s,b,\cdots) \xrightarrow{p_s\to 0} \text{Soft}^{\text{SYM}}(a,s,b)\,\mathcal{A}_{n-1}(\cdots,a,b,\cdots), \tag{4.3}$$

where p_s is the momenta of the soft superfield and a,b are the adjacent superfields. The soft factor $\text{Soft}^{\text{SYM}}(a,s,b)$ is given by [214]

$$\text{Soft}^{\text{SYM}}_{\text{hol}}(a,s,b) = \frac{1}{\varepsilon^2}\text{Soft}(0)^{\text{SYM}}_{\text{hol}}(a,s,b) + \frac{1}{\varepsilon}\text{Soft}(1)^{\text{SYM}}_{\text{hol}}(a,s,b). \tag{4.4}$$

where (0) and (1) indicate the leading and subleading terms. Let us explain the above notation. We associate a pair of spinors $(h_s,\tilde{h}_s)$ with every soft momenta p_s. The limit $(h_s,\tilde{h}_s,\eta^s)\to(\varepsilon h_s,\tilde{h}_s,\eta^s)$ with $\varepsilon\to 0$ and h_s some fixed spinor (namely $h_s\to 0$) is known as the *holomorphic soft limit*. The holomorphic soft factor is then given by [214]

$$\begin{aligned}\text{Soft}(k)^{\text{SYM}}_{\text{hol}}(a,s,b) = \frac{1}{k!}\frac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle}\Bigg[\frac{\langle sa\rangle}{\langle ba\rangle}\left(\tilde{h}^{\dot\alpha}_s\frac{\partial}{\partial\tilde{h}^{\dot\alpha}_b}+(\eta^s)_c\frac{\partial}{\partial(\eta^b)_c}\right)\\ +\frac{\langle sb\rangle}{\langle ab\rangle}\left(\tilde{h}^{\dot\alpha}_s\frac{\partial}{\partial\tilde{h}^{\dot\alpha}_a}+(\eta^s)_c\frac{\partial}{\partial(\eta^a)_c}\right)\Bigg]^k.\end{aligned} \tag{4.5}$$

Similarly the limit $(h_s,\tilde{h}_s,\eta^s)\to(h_s,\varepsilon\tilde{h}_s,\eta^s)$ with $\tilde{h}_s$ a fixed spinor (namely $\tilde{h}_s\to 0$) is known as the *anti-holomorphic* soft limit. The anti-holomorphic soft factor is given by

$$\text{Soft}(k)^{\text{SYM}}_{\text{anti-hol}}(a,s,b) = \frac{1}{k!}\frac{[ab]}{[as][sb]}\delta^4\left(\eta^s+\frac{[as]}{[ab]}\eta^b+\frac{[sb]}{[ab]}\eta^a\right)\left[\frac{[sb]}{[ab]}h^{\alpha}_s\frac{\partial}{\partial h^{\alpha}_a}+\frac{[as]}{[ab]}h^{\alpha}_s\frac{\partial}{\partial h^{\alpha}_b}\right]^k \tag{4.6}$$

The physical soft limit $p_s\to 0$ is equivalent to considering both $h_s,\tilde{h}_s\to 0$ simultaneously. Thus, in the physical soft limit, the soft factor splits as the sum of holomorphic as well as the anti-holomorphic soft factors. We use these results in Section 4.5 to compute soft limits in supergravity.

Next, we discuss the collinear limits. In the collinear limit, we take the momenta of two adjacent superfields p_1 and p_2 to be collinear. Under this limit, the two subfields can fuse to give another superfield with momentum $p_{12} = p_1 + p_2$. We parametrize the momenta of the collinear superfields as

$$p_1 = z p_{12}, \quad p_2 = (1-z) p_{12}, \tag{4.7}$$

where z corresponds to the combined momentum p_{12} on the celestial sphere $\mathcal{CS}^2$. Since $p_1 + p_2 = p_{12}$, we see that, for massless fields, the collinear limit $p_1 || p_2$ implies $p_1 \cdot p_2 \propto p_1^2 = 0$ which is equivalent to the condition $p_{12}^2 \to 0$. Now the collinear limit in $\mathcal{N} = 4$ SYM is given by [126, 221]

$$\mathcal{A}_n(1,2,3,\cdots,n) \overset{p_{12}^2\to 0}{\longrightarrow} \sum_{l=1}^{2} \int d^4\eta^{p_{12}}\, \text{Split}_{1-l}\,(1,2,p_{12})\, \mathcal{A}_{n-1}\,(p_{12},3,\cdots,n)\,. \tag{4.8}$$

The $l = 1,2$ terms in the collinear limits are called the *helicity-preserving* and *helicity-decreasing* processes. The collinear singularity is contained in the split factors. The split factor of the helicity preserving process is given by [126]

$$\text{Split}_0\left(z;\eta^1,\eta^2,\eta^{p_{12}}\right) = \frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle 12\rangle}\prod_{a=1}^{4}\left(\eta_a^{p_{12}} - \sqrt{z}\eta_a^1 - \sqrt{1-z}\eta_a^2\right). \tag{4.9}$$

Whereas for the helicity-decreasing process, the split factor is given by [126]

$$\text{Split}_{-1}\left(z;\eta^1,\eta^2,\eta^{p_{12}}\right) = \frac{1}{\sqrt{z(1-z)}}\frac{1}{[12]}\prod_{a=1}^{4}\left(\eta_a^1\eta_a^2 - \sqrt{1-z}\eta_a^1\eta_a^{p_{12}} + \sqrt{z}\eta_a^2\eta_a^{p_{12}}\right). \tag{4.10}$$

The integral over $\eta^{p_{12}}$ can be performed using general results of Grassmann integration.[1] Here

$$\begin{aligned}\int d\eta_a^{p_{12}}\prod_{a=1}^{4}\left(\eta_a^1\eta_a^2 - \sqrt{1-z}\eta_a^1\eta_a^{p_{12}} + \sqrt{z}\eta_a^2\eta_a^{p_{12}}\right) f\left(\eta_a^{p_{12}}\right)\\ = \delta^{(4)}\left(\sqrt{1-z}\eta_a^1 - \sqrt{z}\eta_a^2\right) f\left(\frac{\eta_a^2}{\sqrt{1-z}}\right),\end{aligned} \tag{4.11}$$

[1] As an example for any function $f(\eta)$ we have [221]

$$\int d^4\eta^p\, \delta^{(4)}\left(\eta^p - \eta\right) f\left(\eta^p\right) = \int d^4\eta^p \prod_{a=1}^{4}\left(\eta_a^p - \eta_a\right) f\left(\eta^p\right) = f(\eta).$$

where, $\delta^{(4)}(\eta^p - \eta) = \prod_{a=1}^{4}\left(\eta_a^p - \eta_a\right)$.

Using these, we express the collinear limit (4.8) as

$$\mathcal{A}_n(1,2,3,\cdots,n) \overset{p_{12}^2\to 0}{\longrightarrow} \frac{1}{\sqrt{z(1-z)}}\frac{1}{[12]}\delta^{(4)}\left(\sqrt{1-z}\eta_a^1-\sqrt{z}\eta_a^2\right)\mathcal{A}_{n-1}\left(\{p_{12},\frac{\eta_a^2}{\sqrt{1-z}}\},3,\cdots,n\right)$$
$$+\frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle 12\rangle}\mathcal{A}_{n-1}(\{p_{12},\sqrt{z}\eta_a^1+\sqrt{1-z}\eta_a^2\},3,\cdots,n). \tag{4.12}$$

Expanding both sides in η^1 and η^2, we can get collinear limits of the component fields. For the collinear limit of component fields, we use the following notation:

$$A_n(1^{h_1},2^{h_2},\ldots,n) \overset{1||2}{\longrightarrow} \sum_h \text{Split}_{-h}^{\text{SYM}}(z,1^{h_1},2^{h_2})A_{n-1}(p^h,\ldots,n), \tag{4.13}$$

where A_n is the amplitude of n different fields in the theory, and the sum is over all helicities in the theory. Note that the split factor is trivial for helicities h, which does not have corresponding interaction with h_1 and h_2. The split also satisfies the conjugation relation [126]

$$\text{Split}_{-h}\left(z;a^{h_1},b^{h_2}\right)=\text{Split}_{+h}\left(z;a^{-h_1},b^{-h_2}\right)|_{[ab]\leftrightarrow\langle ab\rangle} \tag{4.14}$$

The split factor of component fields in SYM has two parts: the kinematic part and the index structure part. The kinematic part only depends on the momenta of the collinear particles, while the index structure consists of the SU(4) R-symmetry indices of the component fields. As indicated earlier, one can compute the kinematic part of the split factors for various combinations of helicities by expanding both sides of (4.12) in η^1,η^2 and then comparing the coefficients. This has been done using Mathematica. The non-trivial split factors for collinear gluons are:

$$\begin{aligned}&\text{Split}_{+1}^{\text{SYM}}\left(z,a^{+1},b^{+1}\right)=0, &&\text{Split}_{-1}^{\text{SYM}}\left(z,a^{+1},b^{+1}\right)=\frac{1}{\sqrt{z(1-z)}}\frac{1}{\langle ab\rangle},\\ &\text{Split}_{+1}^{\text{SYM}}\left(z,a^{-1},b^{+1}\right)=\sqrt{\frac{z^3}{1-z}}\frac{1}{\langle ab\rangle}, &&\text{Split}_{+1}^{\text{SYM}}\left(z,a^{+1},b^{-1}\right)=\frac{(1-z)^2}{\sqrt{z(1-z)}}\frac{1}{\langle ab\rangle}.\end{aligned} \tag{4.15}$$

The split factor for collinear gluinos and scalars are:

$$\begin{aligned}&\text{Split}_{0}^{\text{SYM}}\left(z,a^{+\frac{1}{2}},b^{+\frac{1}{2}}\right)=\frac{1}{\langle ab\rangle}, &&\text{Split}_{+1}^{\text{SYM}}\left(z,a^{+\frac{1}{2}},b^{-\frac{1}{2}}\right)=\frac{(1-z)}{\langle ab\rangle},\\ &\text{Split}_{+1}^{\text{SYM}}\left(z,a^{-\frac{1}{2}},b^{+\frac{1}{2}}\right)=\frac{z}{\langle ab\rangle}, &&\text{Split}_{1}^{\text{SYM}}\left(z,a^{0},b^{0}\right)=\sqrt{z(1-z)}\frac{1}{\langle ab\rangle}.\end{aligned} \tag{4.16}$$

Finally, the split factor for mixed helicities are

$$\begin{aligned}
&\mathrm{Split}^{\mathrm{SYM}}_{+\frac{1}{2}}\left(z,a^{-\frac{1}{2}},b^{+1}\right)=\frac{z}{\sqrt{(1-z)}}\frac{1}{\langle ab\rangle}, \quad \mathrm{Split}^{\mathrm{SYM}}_{+\frac{1}{2}}\left(z,a^{+1},b^{-\frac{1}{2}}\right)=\frac{1-z}{\sqrt{z}}\frac{1}{\langle ab\rangle},\\
&\mathrm{Split}^{\mathrm{SYM}}_{-\frac{1}{2}}\left(z,a^{+\frac{1}{2}},b^{+1}\right)=\frac{1}{\sqrt{(1-z)}}\frac{1}{\langle ab\rangle}, \quad \mathrm{Split}^{\mathrm{SYM}}_{-\frac{1}{2}}\left(z,a^{+1},b^{+\frac{1}{2}}\right)=\frac{1}{\sqrt{z}}\frac{1}{\langle ab\rangle},\\
&\mathrm{Split}^{\mathrm{SYM}}_{0}\left(z,a^{0},b^{+1}\right)=\sqrt{\frac{z}{1-z}}\frac{1}{\langle ab\rangle}, \quad \mathrm{Split}^{\mathrm{SYM}}_{0}\left(z,a^{+1},b^{0}\right)=\sqrt{\frac{1-z}{z}}\frac{1}{\langle ab\rangle},\\
&\mathrm{Split}^{\mathrm{SYM}}_{\frac{1}{2}}\left(z,a^{0},b^{+\frac{1}{2}}\right)=\sqrt{z}\frac{1}{\langle ab\rangle}, \quad \mathrm{Split}^{\mathrm{SYM}}_{\frac{1}{2}}\left(z,a^{+\frac{1}{2}},b^{0}\right)=\sqrt{(1-z)}\frac{1}{\langle ab\rangle}.
\end{aligned} \tag{4.17}$$

All the other split factors can easily be obtained from Eq. (4.14). We now list the index structure part of the split factors for various component fields. We obtain it by expanding both sides of (4.12) in η^1 and η^2 and comparing the coefficients. Some of the index structures have been worked out in [221]. We complete the list here. Note that the index structure in the collinear limit of a gluon with any other component field is trivially determined; hence, we omit them from the table below.

Collinear fields	Resulting index structure
Γ^a_+, Γ^b_+	Φ^{ab}
Γ^-_a, Γ^-_b	$\frac{1}{2!}\varepsilon_{abcd}\Phi^{cd}$
Γ^a_+, Γ^-_b	$\delta^a_b G^{\pm}$
Γ^a_+, Φ^{bc}	$\varepsilon^{abcd}\Gamma^-_d$
Γ^-_a, Φ^{bc}	$2!\delta^{[b}_a\Gamma^{c]}_+$
Φ^{ab}, Φ^{cd}	$\varepsilon^{abcd}G^{\pm}$

Table 4.1: Index structure in collinear limit in $\mathcal{N}=4$ SYM

4.3 $\mathcal{N}$=8 Supergravity

In this section, we briefly review the field contents and basic properties of the theory and also establish notations that we follow in the remainder of the chapter.

Let $\{\eta_A\}_{A=1}^8$ be the Grassmann coordinates on the $\mathcal{N}=8$ superspace. The degrees of $\mathcal{N}=8$ supergravity for an on-shell superfield is defined as

$$\Psi(p,\eta) = H^+(p) + \eta_A \psi_+^A(p) + \eta_{AB} G_+^{AB}(p) + \eta_{ABC}\chi_+^{ABC}(p) \\ + \eta_{ABCD}\Phi^{ABCD}(p) + \tilde{\eta}^{ABC}\chi_{ABC}^-(p) + \tilde{\eta}^{AB}G_{AB}^-(p) + \tilde{\eta}^A\psi_A^-(p) + \tilde{\eta}H^-(p), \tag{4.18}$$

where we have introduced the notation

$$\begin{aligned} \eta_{A_1\ldots A_n} &\equiv \frac{1}{n!}\eta_{A_1}\cdots\eta_{A_2} \\ \tilde{\eta}^{A_1\ldots A_n} &\equiv \varepsilon^{A_1\ldots A_n B_1\ldots B_{8-n}}\eta_{B^1\ldots B^{8-n}} \\ \tilde{\eta} &\equiv \prod_{A=1}^{8}\eta_A. \end{aligned} \tag{4.19}$$

The fields $H^\pm$ represent graviton, G_+^{AB} and G_{AB}^- represent gluons, ψ_+^A and ψ_A^- represent gravitinos, χ_+^{ABC} and χ_{ABC}^- represent gluinos and finally Φ^{ABCD} represent the real scalars. The (sub)super scripts $\pm$ denote the positive and negative helicity of various fields. The superamplitude is then defined by

$$\mathcal{M}_n(\{p_1,\eta^1\},\ldots\{p_n,\eta^n\}) = \langle\Psi_1(p_1,\eta^1)\ldots\Psi_n(p_n,\eta^n)\rangle. \tag{4.20}$$

We now explain the factorization of states in supergravity into tensor product of states in super Yang-Mills. We begin by counting the degrees of freedom in the two theories. It is summarised in Table 4.2 below.

The precise factorization of fields and operators are given in [215]. We summarise the factorization in Table 4.3 below. The second factor of $\mathcal{N}=4$ SYM is written with a tilde to emphasize that the factors of the two gauge theories are not identical. The following notation is used in the table below and in the rest of the chapter: uppercase indices $A,B,C,D,\ldots \in \{1,\ldots,8\}$ will denote indices in $\mathcal{N}=8$ supergravity, lower case indices $a,b,c,d \in \{1,2,3,4\}$ correspond to first SYM factor and $r,s,t,u \in \{5,6,7,8\}$ correspond to second SYM factor. In particular, in equations where both upper and lower case indices have been used, we will

$\mathcal{N}=8$ Supergravity	$(\mathcal{N}=4\text{ SYM})\otimes(\mathcal{N}=4\text{ SYM})$
70 Scalars	$36(0\otimes 0);1(-1\otimes +1);1(+1\otimes -1);16(+\frac{1}{2}\otimes -\frac{1}{2});16(-\frac{1}{2}\otimes +\frac{1}{2})$
112 Gravi-photinos $(\pm)$	$48(\pm\frac{1}{2}\otimes 0);48(0\otimes \pm\frac{1}{2});8(\pm\frac{1}{2}\otimes \mp 1);8(\pm 1\otimes \mp\frac{1}{2})$
56 Graviphotons $(\pm)$	$12(\pm 1\otimes 0);16(+\frac{1}{2}\otimes +\frac{1}{2});16(-\frac{1}{2}\otimes -\frac{1}{2})$
16 Gravitinos $(\pm)$	$8(\pm\frac{1}{2}\otimes \pm 1);8(\pm 1\otimes \pm\frac{1}{2})$
2 Gravitons $(\pm)$	$2(\pm 1\otimes \pm 1)$

Table 4.2: Factorisation of $\mathcal{N}=8$ supergravity states into $\mathcal{N}=4$ super Yang-Mills states

assume $A=a$ and $A=r$ and so on when $1\leq A\leq 4$ and $5\leq A\leq 8$, respectively. Further note that the scalars in supergravity and super Yang-Mills satisfy the self duality relation, [215]

$$\begin{aligned}\Phi_{ABCD} &= \frac{1}{4!}\alpha_8\varepsilon_{ABCDEFGH}\Phi^{EFGH}\\ \Phi_{ab} &= \frac{1}{2!}\alpha_4\varepsilon_{abcd}\Phi^{cd}\\ \tilde{\Phi}_{rs} &= \frac{1}{2!}\tilde{\alpha}_4\varepsilon_{rstu}\Phi^{tu}\end{aligned} \tag{4.21}$$

with $\alpha_4,\tilde{\alpha}_4,\alpha_8\in\{\pm 1\}$ along with the consistency condition [215, Eq. 2.12],

$$\alpha_4\tilde{\alpha}_4=\alpha_8. \tag{4.22}$$

and $\varepsilon_{AB\ldots H}$ is the Levi-Civita tensor in 8 dimensions and $\varepsilon_{abcd},\varepsilon_{rstu}$ are Levi-Civita tensor in 4 dimensions. Note that since $5\leq r,s,t,u\leq 8$, ε_{rstu} is defined using permutations of $5,6,7,8$. Using this factorization, we can find the collinear limit of any two states in $\mathcal{N}=8$ supergravity. The possible choices of the self-duality factors $(\alpha_4,\tilde{\alpha}_4,\alpha_8)$ are $(1,1,1),(1,-1,-1),(-1,1,-1),(-1,-1,1)$. Based on possible choices of the self-duality factors, we have four different ways of getting the supergravity amplitudes via double copy.

4.4 Collinear limits in $\mathcal{N}=8$ supergravity

In this section, we compute the collinear limits using the component field formalism. The double copy relation of collinear limits in component formalism is given by

$$M_n(1^{h_1},2^{h_2},\ldots,n)\xrightarrow{1||2}\sum_h \text{Split}^{\text{SG}}_{-h}(z,1^{h_1},2^{h_2})M_{n-1}(p^h,\ldots,n), \tag{4.23}$$

$H^+ = G^+\tilde{G}^+$	$H^- = G^-\tilde{G}^-$
$\Psi^a_+ = \Gamma^a_+\tilde{G}^+$ $\Psi^r_+ = G^+\tilde{\Gamma}^r_+$	$\Psi^-_a = \Gamma^-_a\tilde{G}^-$ $\Psi^-_r = G^-\tilde{\Gamma}^-_r$
$G^+_{ab} = \Phi^{ab}\tilde{G}^+$ $G^+_{ar} = \Gamma^a_+\tilde{\Gamma}^r_+$ $G^+_{rs} = G^+\tilde{\Phi}^{rs}$	$G^-_{ab} = \Phi_{ab}\tilde{G}^-$ $G^-_{ar} = -\Gamma^-_a\tilde{\Gamma}^-_r$ $\bar{G}^-_{rs} = G^-\tilde{\Phi}_{rs}$
$\chi^{abc}_+ = \alpha_4\varepsilon^{abcd}\Gamma^-_d\tilde{G}^+$ $\chi^{abr}_+ = \Phi^{ab}\tilde{\Gamma}^r_+$ $\chi^{ars}_+ = \Gamma^a_+\tilde{\Phi}^{rs}$ $\chi^{rst}_+ = \tilde{\alpha}_4\varepsilon^{rstu}G^+\tilde{\Gamma}^-_u$	$\chi^-_{abc} = -\alpha_4\varepsilon_{abcd}\Gamma^d_+\tilde{G}^-$ $\chi^-_{abr} = \Phi_{ab}\tilde{\Gamma}^-_r$ $\chi^-_{ars} = \Gamma^-_a\tilde{\Phi}_{rs}$ $\chi^-_{rst} = -\tilde{\alpha}_4\varepsilon_{rstu}G^-\tilde{\Gamma}^u_+$
$\Phi^{abcd} = \alpha_4\varepsilon^{abcd}G^-\tilde{G}^+$ $\Phi^{abcr} = \alpha_4\varepsilon^{abcd}\Gamma^-_d\tilde{\Gamma}^r_+$ $\Phi^{abrs} = \Phi^{ab}\tilde{\Phi}^{rs}$ $\Phi^{arst} = \tilde{\alpha}_4\varepsilon^{rstu}\Gamma^a_+\tilde{\Gamma}^-_u$ $\Phi^{rstu} = \tilde{\alpha}_4\varepsilon^{rstu}G^+\tilde{G}^-$	$\Phi_{abcd} = \alpha_4\varepsilon_{abcd}G^+\tilde{G}^-$ $\Phi_{abcr} = \alpha_4\varepsilon_{abcd}\Gamma^d_+\tilde{\Gamma}^-_r$ $\Phi_{abrs} = \Phi_{ab}\tilde{\Phi}_{rs}$ $\Phi_{arst} = \tilde{\alpha}_4\varepsilon_{rstu}\Gamma^-_a\tilde{\Gamma}^u_+$ $\Phi_{rstu} = \tilde{\alpha}_4\varepsilon_{rstu}G^-\tilde{G}^+$

Table 4.3: Factorisation of states in supergravity into states in super Yang-Mills

where the split factor $\text{Split}^{\text{SG}}_{-h}(z,1^{h_1},2^{h_2})$ is given in terms of the split factors in $\mathcal{N}=4$ super Yang-Mills theory as follows:

$$\begin{aligned}\text{Split}^{\text{SG}}_{-(h+\tilde{h})}\left(z,1^{h_1+\tilde{h}_1},2^{h_2+\tilde{h}_2}\right) = -s_{12}\times\text{Split}^{\text{SYM}}_{-h}\left(z,1^{h_1},2^{h_2}\right)\\ \times\text{Split}^{\text{SYM}}_{-\tilde{h}}\left(z,2^{\tilde{h}_2},1^{\tilde{h}_1}\right)\end{aligned} \tag{4.24}$$

where $(h+\tilde{h})$ is the factorisation of $\mathcal{N}=8$ supergravity state with total spin $h+\tilde{h}$ in terms of two copies of $\mathcal{N}=4$ super Yang-Mills states with spins $h,\tilde{h}$ respectively according to Table 4.3 and $s_{12}=\langle 12\rangle[21]$. The sum over all $\mathcal{N}=8$ supergravity states is interpreted as a double sum over a tensor product of $\mathcal{N}=4$ SYM states [216]. The calculation of collinear limit then involves two steps:

1. Calculate the split factors for all possible factorization channels, that is, for all possible values of spin and helicity states h in $\mathcal{N}=8$ supergravity. This can be done in such that the factorization $h=h_1+h_2$ into different spin and helicity states in $\mathcal{N}=4$ SYM from Table 4.3 gives nontrivial split factors. In general, one only needs to calculate half of all possible combinations of helicities. The remaining split factors can be calculated using

$$\text{Split}_-\left(z;a^{h_1},b^{h_2}\right)=\text{Split}_+\left(z;a^{-h_1},b^{-h_2}\right)|_{[ab]\leftrightarrow\langle ab\rangle} \tag{4.25}$$

2. Write the collinear limit of amplitudes by consistently matching the R-symmetry factors using Table 4.1, which is non-trivial in the case of $\mathcal{N}>1$ theories.

4.4.1 Collinear limits of like spins

Here, we compute the collinear amplitudes from the splits for states of the same spin. We will show the computation for some cases and summarise the results for the rest in tabular form and refer the reader to Appendix B.2.1 for all the details of the computations. Moreover, we only summarise the collinear limits for independent cases not related by Eq.(4.25).

***Gravitons*:**

When both collinear gravitons are of the same helicity (positive or negative), then from Table

$M_n\left(1^{+2},2^{+2},\cdots,n\right)$	$\frac{\omega_p^2}{\omega_1\omega_2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{+2},\ldots,n\right)$
$M_n\left(1^{-2},2^{-2},\cdots,n\right)$	$\frac{\omega_p^2}{\omega_1\omega_2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{-2},\ldots,n\right)$
$M_n\left(1^{+2},2^{-2},\ldots,n\right)$	$\frac{\omega_1^3}{\omega_p^2\omega_2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{-2},3,\ldots,n\right)+\frac{\omega_2^3}{\omega_p^2\omega_1}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{+2},3,\ldots,n\right)$

Table 4.4: Amplitude corresponding to two collinear gravitons

4.3, we see that

$$\begin{aligned}\text{Split}^{\text{SG}}_{-(h+\tilde{h})}\left(z,1^{\pm2},2^{\pm2}\right)=-s_{12}\times\text{Split}^{\text{SYM}}_{-h}\left(z,1^{\pm1},2^{\pm1}\right)\\ \times\text{Split}^{\text{SYM}}_{-\tilde{h}}\left(z,2^{\pm1},1^{\pm1}\right).\end{aligned}$$

A similar factorization is true for opposite helicities. Thus split factors in $\mathcal{N}=8$ supergravity for two collinear gravitons is

$$\begin{aligned}&\text{Split}^{\text{SG}}_{+2}\left(z,a^{+2},b^{+2}\right)=0=\text{Split}^{\text{SG}}_{-2}\left(z,a^{-2},b^{-2}\right),\\ &\text{Split}^{\text{SG}}_{-2}\left(z,a^{+2},b^{+2}\right)=-\frac{1}{z(1-z)}\frac{[ab]}{\langle ab\rangle},\quad \text{Split}^{\text{SG}}_{+2}\left(z,a^{-2},b^{-2}\right)=-\frac{1}{z(1-z)}\frac{\langle ab\rangle}{[ab]}\\ &\text{Split}^{\text{SG}}_{+2}\left(z,a^{-2},b^{+2}\right)=-\frac{z^3}{(1-z)}\frac{[ab]}{\langle ab\rangle},\quad \text{Split}^{\text{SG}}_{-2}\left(z,a^{-2},b^{+2}\right)=-\frac{(1-z)^3}{z}\frac{\langle ab\rangle}{[ab]}.\end{aligned}\tag{4.26}$$

Writing the momenta of the collinear particles as $p_i=\omega_i q_i$, $i=1,2$, the momenta along the collinear channel is $p=p_1+p_2=\omega_p q_p$ with $\omega_p=\omega_1+\omega_2$ and we can write

$$p_1=zp,\quad p_2=(1-z)p.\tag{4.27}$$

Note that $q_p=q_1=q_2$ and hence

$$z=\frac{\omega_1}{\omega_p},\quad (1-z)=\frac{\omega_2}{\omega_p}$$

With this parametrization, the collinear limits can be tabulated as, Here in LHS, $1,2,\ldots,n$ refers to external particles with momenta $p_1,p_2,\ldots,p_n$, and p_1 is taken collinear to p_2 according to the parametrization in Eq. (4.27). We will carry this notation throughout the chapter.

Note that the collinear limit of two negative helicity gravitons from the collinear limit of two positive helicity gravitons by flipping the helicities throughout and $z_{12}\leftrightarrow\bar{z}_{12}$. This is reminiscent of Eq.(4.25).

***Gravitinos*:**

The non-trivial split factors in $\mathcal{N}=8$ Supergravity for two collinear gravitinos are given by

$$\begin{aligned}
&\text{Split}^{\text{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+1}\right)=-\frac{1}{\sqrt{z(1-z)}}\frac{[12]}{\langle 12\rangle}, \quad \text{Split}^{\text{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{1+\frac{1}{2}}\right)=-\frac{1}{\sqrt{z(1-z)}}\frac{[12]}{\langle 12\rangle}\\
&\text{Split}^{\text{SG}}_{-2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right)=-\sqrt{\frac{z^5}{(1-z)}}\frac{\langle 12\rangle}{[12]}, \quad \text{Split}^{\text{SG}}_{+2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right)=-\sqrt{\frac{(1-z)^5}{z}}\frac{[12]}{\langle 12\rangle}
\end{aligned} \tag{4.28}$$

We can calculate other split factors using Eq.(4.25). The factorization of R-symmetry indices has the form.

$$\begin{cases} \left(a;\frac{3}{2}\right)=\left(a;\frac{1}{2}\right)\otimes 1\\ \left(r;\frac{3}{2}\right)=1\otimes\left(r;\frac{1}{2}\right)\end{cases}$$

Corresponding to the above two factorization the amplitudes following Eq.(4.23) and Eq.(4.24), can be combined and written in the table below. For details, we refer the readers to section B.2.1 in the Appendix.

$M_n\left(1^{A;+\frac{3}{2}},2^{B;+\frac{3}{2}},\cdots,n\right)$	$\frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{AB;+1},\cdots,n\right)$
$M_n\left(1^{A;+\frac{3}{2}},2_B^{-\frac{3}{2}},\cdots,n\right)$	$\delta^A_B\frac{\omega_2^{\frac{5}{2}}}{\omega_1^{\frac{1}{2}}\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{-2},\cdots,n\right)+\delta^A_B\frac{\omega_1^{\frac{5}{2}}}{\omega_2^{\frac{1}{2}}\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{+2},\cdots,n\right)$

Table 4.5: Amplitude corresponding to two collinear gravitinos

***Gravi-photons*:**

Using the factorisation in Eq.(4.24) the non-trivial split factors for two collinear Graviphotons is given in Appendix B.1. The calculation of collinear limits is done in Appendix B.2.1. The result is recorded in the table below.

The R-symmetry index factorizes as follows:

$$\begin{cases} (ab;1)=(ab;0)\otimes 1\\ (ar;1)=\left(a,\frac{1}{2}\right)\otimes\left(r;\frac{1}{2}\right)\\ (rs;1)=1\otimes(rs;0)\end{cases}$$

From the above factorisation we can combine all of the non-trivial amplitudes for $1\leq A,B\leq 8$

as,

$M_n\left(1^{AB;+1},2^{CD;+1},\cdots\right)$	$\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ABCD;0},\cdots\right)$
$M_n\left(1^{AB;+1},2^{-1}_{CD},\cdots\right)$	$-\delta^{AB}_{CD}\left[\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-2},\cdots\right)+\frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{+2},\cdots\right)\right]$

Table 4.6: Amplitude corresponding to two collinear graviphotons

In writing the collinear limit of opposite helicity graviphotons, we made a choice of self-duality factors $\alpha_4=\tilde{\alpha}_4=-1,\alpha_8=1$. This choice is unique and motivated by our aim to make the R-symmetry indices consistent in both sides of the amplitude calculations. See Appendix B.2 for details.

***Graviphotinos*:**

Following the factorisation in Eq. (4.24), the non-trivial split factors for this channel in $\mathcal{N}=8$ supergravity are given in Appendix B.2.

The factorisation of the R-symmetry indices is as follows:

$$\begin{cases}\left(abr;\frac{1}{2}\right)=(ab;0)\otimes\left(r;\frac{1}{2}\right)\\ \left(ars;\frac{1}{2}\right)=\left(a;\frac{1}{2}\right)\otimes(rs;0)\end{cases}$$

$$\begin{cases}\left(rst;\frac{1}{2}\right)=-\varepsilon^{rstu}\left(1\otimes\left(u;-\frac{1}{2}\right)\right)\\ \left(abc;\frac{1}{2}\right)=-\varepsilon^{abcd}\left(\left(d;-\frac{1}{2}\right)\otimes 1\right)\end{cases}\qquad(\text{sum over } u,d)$$

The amplitudes corresponding to the above factorisation channels are summarised as follows

$M_n\left(1^{ars;+\frac{1}{2}},2^{btu;+\frac{1}{2}},\cdots\right)$	$\varepsilon^{rstu}\varepsilon^{abcd}\ \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{cd},\cdots\right)$
$M_n\left(1^{ars;+\frac{1}{2}},2^{bct;+\frac{1}{2}},\cdots\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\ \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{du},\cdots\right)$
$M_n\left(1^{rst;+\frac{1}{2}},2^{abc;+\frac{1}{2}},\cdots\right)$	$\varepsilon^{rstu}\varepsilon^{abcd}\ \frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{ud},\cdots\right)$
$M_n\left(1^{ars;+\frac{1}{2}},2^{-\frac{1}{2}}_{btu},\cdots\right)$	$\varepsilon_{tuvw}\varepsilon^{rsvw}\delta^a_b\left[\frac{\omega_1^{\frac{3}{2}}\omega_2^{\frac{1}{2}}}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{+2},\cdots\right)+\frac{\omega_2^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{-2},\cdots\right)\right]$

Table 4.7: Amplitude corresponding to two collinear graviphotinos

***Scalars*:**

The three possible channels are $0 = 0 \otimes 0$, $0 = \pm 1 \otimes \mp 1$ and $0 = \pm\frac{1}{2} \otimes \mp\frac{1}{2}$. We have the non-trivial splits given in Appendix B.3.

The factorizations of R-symmetry indices are given by

$$(abrs;0) = (ab;0) \otimes (rs;0)$$

$$\begin{cases} (abcd;0) = -\varepsilon^{abcd}(-1 \otimes 1) \\ (rstu;0) = -\varepsilon^{rstu}(1 \otimes -1) \end{cases}$$

$$\begin{cases} (abcr;0) = -\varepsilon^{abcd}\left(d;-\frac{1}{2}\right) \otimes \left(r;\frac{1}{2}\right) \\ (arst;0) = -\varepsilon^{rstu}\left(a;\frac{1}{2}\right) \otimes \left(u;-\frac{1}{2}\right) \end{cases}$$

The factorised amplitudes are,

$M_n\left(1^{abrs;0}, 2^{cdtu;0}, \cdots\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\left[\frac{\omega_1\omega_2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2},\cdots\right) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2},\cdots\right)\right]$
$M_n\left(1^{abcd;0}, 2^{rstu;0}, \cdots\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2},\cdots\right) \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2},\cdots\right)\right]$
$M_n\left(1^{abcu;0}, 2^{drst;0}, \cdots\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2},\cdots\right) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2},\cdots\right)\right]$
$M_n\left(1^{arst;0}, 2^{bcdu;0}, \cdots\right)$	$\varepsilon^{rstu}\varepsilon^{abcd}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2},\cdots\right) + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2},\cdots\right)\right]$

Table 4.8: Amplitude corresponding to two collinear scalars

4.4.2 Collinear limits of Mixed Spins

In this section, we list the collinear limit of states with different spins. The non-trivial split factors are listed in Appendix B.1 and the detailed calculation is done in Appendix B.2.2.

***Graviton-Gravitino*:**

The non-trivial split factors for this collinear pair are given in Appendix B.4.

Using different factorisation channels of the Gravitinos we have,

$M_n\left(1^{+2},2^{r;+\frac{3}{2}},\cdots,n\right)$	$\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{r;+\frac{3}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2_r^{-\frac{3}{2}},\cdots,n\right)$	$\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{a;+\frac{3}{2}},\cdots,n\right)$	$\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{a;+\frac{3}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2_a^{-\frac{3}{2}},\cdots,n\right)$	$\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_a^{-\frac{3}{2}},\cdots,n\right)$

Table 4.9: Amplitude corresponding to collinear graviton and gravitino

***Graviton-Graviphoton*:**

The split factors are given in Appendix B.5. For $1\leq A,B\leq 8$ all the amplitudes corresponding to different factorisation channels are summarised as

$$M_n\left(1^{+2},2_{AB}^{-1},\cdots,n\right)=\frac{\omega_2^2}{\omega_1\,\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{AB}^{-1},\cdots,n\right)$$

***Graviton-Graviphotino*:**

Non-trivial split factors are given in Appendix B.6.

$M_n\left(1^{+2},2^{abr;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abr;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{ars;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ars;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{abc;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abc;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2^{rst;+\frac{1}{2}},\cdots,n\right)$	$\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{rst;+\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2_{abr}^{-\frac{1}{2}},\cdots,n\right)$	$\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{abr}^{-\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2_{ars}^{-\frac{1}{2}},\cdots,n\right)$	$\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{ars}^{-\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2_{abc}^{-\frac{1}{2}},\cdots,n\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{abc}^{-\frac{1}{2}},\cdots,n\right)$
$M_n\left(1^{+2},2_{rst}^{-\frac{1}{2}},\cdots,n\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{rst}^{-\frac{1}{2}},\cdots,n\right)$

Table 4.10: Amplitude corresponding to collinear graviton and graviphotino

***Graviton-Scalar*:**

The non-trivial split factors are given in Appendix B.7. For all the factorisation channel for the Scalars in $\mathcal{N}=8$ the split factors will remain the same. Hence

$$\boxed{M_n\left(1^{+2},2^{ABCD;0},\cdots,n\right)=\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ABCD;0},\cdots,n\right)}$$

***Gravitino-Graviphoton*:**

The split factors for this collinear pair are given in Appendix B.8. For any $1\leq A,B\leq 8$ we have

$$\boxed{M_n\left(1^{A;+\frac{3}{2}},2^{-1}_{BC},\cdots,n\right)=\frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\,\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}2!\delta^A_{[B}\times M_{n-1}\left(p^{-\frac{3}{2}}_{C]},\cdots,n\right)}$$

Here [...] indicates antisymmetrized indices defined by

$$p_{[A_1\ldots A_n]}:=\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sign}(\sigma)\;p_{A_{\sigma(1)}\ldots A_{\sigma(n)}}. \tag{4.29}$$

***Gravitino-Graviphotino*:**

The split factors are given in Appendix B.9.

$M_n\left(1^{A;+\frac{3}{2}},2^{BCD;+\frac{1}{2}},\cdots\right)$	$\sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{ABCD;0},\cdots\right)$
$M_n\left(1^{A;+\frac{3}{2}},2^{-\frac{1}{2}}_{BCD},\cdots\right)$	$-\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times 3\delta^A_{(B}M_{n-1}\left(p^{-1}_{CD)},\cdots\right)$

Table 4.11: Amplitude corresponding to collinear gravitino and graviphotino

Here (...) indicates symmetrized indices defined by

$$p_{(A_1\ldots A_n)}:=\frac{1}{n!}\sum_{\sigma\in S_n}p_{A_{\sigma(1)}\ldots A_{\sigma(n)}}. \tag{4.30}$$

***Gravitino-scalar*:**

The splits are given in Appendix B.10.

$$\boxed{M_n\left(1^{A;+\frac{3}{2}},2^{BCDE;0},\cdots,n\right)=-\frac{1}{3!}\varepsilon^{ABCDEFGH}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{FGH},\cdots,n\right)}$$

***Graviphoton-Graviphotino*:**

The splits are in Appendix B.11.

$M_n\left(1^{ab;+1}, 2^{cdr;+\frac{1}{2}}, \cdots, n\right)$	$\sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}\left(p^{ABCD;0}, \cdots\right)$
$M_n\left(1^{ab;+1}, 2_{cdr;-\frac{1}{2}}, \cdots, n\right)$	$-\delta^{ab}_{cd}\,\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_r^{-\frac{3}{2}}, \cdots, n\right)$

Table 4.12: Amplitude corresponding to collinear graviphoton and graviphotino

***Graviphoton-scalar*:**

The splits for this collinear pair are in Appendix B.12.

$M_n\left(1^{ab;+1}, 2^{cdrs;0}, \cdots, n\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{tu}^{-1}, \cdots, n\right)$
$M_n\left(1^{rs;+1}, 2^{abtu;0}, \cdots, n\right)$	$\varepsilon^{rstu}\varepsilon^{abcd}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{cd}^{-1}, \cdots, n\right)$
$M_n\left(1^{ar;+1}, 2^{bcst;0}, \cdots, n\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{du}^{-1}, \cdots, n\right)$
$M_n\left(1^{ab;+1}, 2^{cdef;0}, \cdots, n\right)$	$-\varepsilon^{cdef}\varepsilon^{abgh}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{gh}^{-1}, \cdots, n\right)$
$M_n\left(1^{rs;+1}, 2^{cdef;0}, \cdots, n\right)$	$-\varepsilon^{cdef}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{tu}^{-1}, \cdots, n\right)$
$M_n\left(1^{ar;+1}, 2^{bcds;0}, \cdots, n\right)$	$-\varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{tu}^{-1}, \cdots, n\right)$
$M_n\left(1_{ab}^{-1}, 2^{cdef;0}, \cdots, n\right)$	$-\varepsilon^{cdef}\varepsilon_{abgh}\,\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{gh;+1}, \cdots, n\right)$
$M_n\left(1_{ar}^{-1}, 2^{bcds;0}, \cdots, n\right)$	$-\varepsilon^{bcde}\delta^s_r\varepsilon_{aefg}\,\frac{\omega_1}{\omega_p}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{fg;+1}, \cdots, n\right)$
$M_n\left(1_{ar}^{-1}, 2^{bcst;0}, \cdots, n\right)$	$-\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}4!\delta_a^{[b}\delta_r^s\, M_{n-1}\left(p^{tc];+1}, \cdots, n\right)$

Table 4.13: Amplitude corresponding to collinear graviphoton and scalar

***Graviphotino-scalar*:**

The splits are given in Appendix B.13.

$M_n\left(1^{abr;+\frac{1}{2}}, 2^{cdst;0}, \cdots, n\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_u^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{abr;+\frac{1}{2}}, 2^{cstu;0}, \cdots, n\right)$	$-\varepsilon^{abcd}\varepsilon^{rstu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_d^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1^{ars;+\frac{1}{2}}, 2^{bctu;0}, \cdots, n\right)$	$\varepsilon^{abcd}\varepsilon^{rstu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_d^{-\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{ars}^{-\frac{1}{2}}, 2^{bctu;0}, \cdots, n\right)$	$-2!\delta_{rs}^{tu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}}\delta_a^{[b} \times M_{n-1}\left(p^{c];+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{ars}^{-\frac{1}{2}}, 2^{btuv\ 0}, \cdots, n\right)$	$-\delta_a^b\varepsilon^{tuvw}\varepsilon_{wrsx}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{x;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{avwx;0}, \cdots, n\right)$	$-\varepsilon_{rstu}\varepsilon^{vwxu}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)$
$M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{uvwx;0}, \cdots, n\right)$	$-\varepsilon_{rsty}\varepsilon^{uvwx}\ \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{y;+\frac{3}{2}}, \cdots, n\right)$

Table 4.14: Amplitude corresponding to collinear graviphotino and scalar

4.5 Soft Limits in $\mathcal{N}$=8 supergravity

To complete our study, we now move on to study the soft limit of supergravity amplitudes. In particular, in this section, we will compute the soft limits of graviton and gravitinos up to sub-subleading order. As explained earlier, for both holomorphic and antiholomorphic soft limits for supergravity amplitudes, we have,

$$\begin{aligned}
\mathcal{M}_{n+1}\left(\Psi_s, \Psi_1, \ldots, \Psi_n\right) &\overset{\varepsilon \to 0}{=} \sum_{k=0}^{2} \frac{1}{\varepsilon^{3-k}} \mathcal{S}^{(k)} \mathcal{M}_n\left(\Psi_1, \ldots, \Psi_n\right) \quad \text{(holomorphic soft limit)} \\
\mathcal{M}_{n+1}\left(\Psi_s, \Psi_1, \ldots, \Psi_n\right) &\overset{\varepsilon \to 0}{=} \sum_{k=0}^{2} \frac{1}{\varepsilon^{3-k}} \overline{\mathcal{S}}^{(k)} \mathcal{M}_n\left(\Psi_1, \ldots, \Psi_n\right) \quad \text{(antiholomorphic soft limit)}
\end{aligned} \tag{4.31}$$

where in both cases the holomorphic and antiholomorphic soft limits are parametrized by $\varepsilon \to 0$ for the soft superfield Ψ_s and $\mathcal{S}^{(k)}$ and $\overline{\mathcal{S}}^{(k)}$ are soft operators corresponding to these limits.

4.5.1 Graviton soft limit

Recall that in the physical soft limit $p_s \to 0$ or equivalently $h_s, \tilde{h}_s \to 0$, the leading soft factor in SYM is given by the sum of leading soft factors in holomorphic and anti-holomorphic soft limit:

$$\text{Soft}^{\text{SYM}}_{\text{leading}}(a,s,b) = \frac{\langle ab\rangle}{\langle as\rangle\langle sb\rangle} + \frac{[ab]}{[as][sb]}\delta^4(\eta^s) \tag{4.32}$$

As described in Section 1.5.1, we will use the double copy relation [113, Eq. 2.15]

$$\begin{aligned}&\frac{1}{\varepsilon^3}\text{Soft}(0)^{\text{SG}} + \frac{1}{\varepsilon^2}\text{Soft}(1)^{\text{SG}} + \frac{1}{\varepsilon}\text{Soft}(2)^{\text{SG}} \\ &\qquad = \sum_{i=1}^{n} \varepsilon\langle si\rangle[is]\left(\frac{1}{\varepsilon^2}\text{Soft}(0)^{\text{SYM}}(i,s,a) + \frac{1}{2\varepsilon}\text{Soft}(1)^{\text{SYM}}(i,s,a)\right)^2.\end{aligned} \tag{4.33}$$

Comparing the coefficients of ε powers, we get

$$\begin{aligned}\text{Soft}(0)^{\text{SG}} &= \sum_{i=1}^{n}\langle si\rangle[is]\left[\text{Soft}(0)^{\text{SYM}}(i,s,a)\right]^2 \\ \text{Soft}(1)^{\text{SG}} &= \sum_{i=1}^{n}\langle si\rangle[is]\left[\text{Soft}(0)^{\text{SYM}}(i,s,a)\times\text{Soft}(1)^{\text{SYM}}(i,s,a)\right] \\ \text{Soft}(2)^{\text{SG}} &= \frac{1}{4}\sum_{i=1}^{n}\langle si\rangle[is]\left[\text{Soft}(1)^{\text{SYM}}(i,s,a)\right]^2\end{aligned} \tag{4.34}$$

Thus, the double copy relation gives the sum of leading, subleading, and sub-subleading soft factors in supergravity in terms of the leading and subleading soft factors in SYM. It is clear that the leading and subleading soft factors in supergravity are given by

$$\text{Soft}^{\text{SG}}_{\text{leading}} = \sum_{i=1}^{n}\langle si\rangle[is]\left([\text{Soft}(0)^{\text{SYM}}_{\text{hol}}(i,s,a)]^2 + [\text{Soft}(0)^{\text{SYM}}_{\text{anti-hol}}(i,s,a)]^2\right) \tag{4.35}$$

and

$$\begin{aligned}\text{Soft}^{\text{SG}}_{\text{subleading}} = \sum_{i=1}^{n}\langle si\rangle[is]\Big[&\text{Soft}(0)^{\text{SYM}}_{\text{hol}}(i,s,a)\times\text{Soft}(1)^{\text{SYM}}_{\text{hol}}(i,s,a) \\ &+\text{Soft}(0)^{\text{SYM}}_{\text{anti-hol}}(i,s,a)\times\text{Soft}(1)^{\text{SYM}}_{\text{anti-hol}}(i,s,a)\Big]\end{aligned} \tag{4.36}$$

We now substitute (4.5) and (4.6) into (4.35) and (4.36) to get the leading and subleading soft factors in supergravity. Note that the nonholomorphic soft factor in SYM includes the Grassmann delta function $\delta^4(\eta)$. So, while squaring the nonholomorphic soft factor of SYM, the square of this delta function in double copy is interpreted as the Grassmann delta function on $\mathcal{N}=8$ superspace:

$$\left(\delta^4(\eta_a)\right)^2 = \delta^8(\eta_A) \tag{4.37}$$

where the indices have the usual meanings with a running from 1 to 4 and A running from 1 to 8. The leading soft factor is then given by

$$\text{Soft}^{\text{SG}}_{\text{leading}}(a,s,b) = \sum_{i=1}^{n}\left(\frac{[si]}{\langle si\rangle}\frac{\langle ai\rangle^2}{\langle as\rangle^2} + \frac{\langle si\rangle}{[si]}\frac{[ai]^2}{[as]^2}\delta^8(\eta_A)\right) \tag{4.38}$$

We now evaluate the subleading soft limit. From (4.6) and (4.36) we have,

$$\begin{aligned}
&\text{Soft}^{\text{SG}}_{\text{subleading}} \\
&= \sum_{i=1}^{n}\langle si\rangle[is]\left[\frac{\langle ia\rangle}{\langle is\rangle\langle sa\rangle}\left\{\frac{1}{\langle sa\rangle}\left(\tilde{h}^{\dot\alpha}_s\frac{\partial}{\partial\tilde{h}^{\dot\alpha}_a} + \eta^s_A\frac{\partial}{\partial\eta^a_A}\right) + \frac{1}{\langle is\rangle}\left(\tilde{h}^{\dot\alpha}_s\frac{\partial}{\partial\tilde{h}^{\dot\alpha}_i} + \eta^A_s\frac{\partial}{\partial\eta^A_i}\right)\right\}\right. \\
&\left. + \frac{[ia]}{[is][sa]}\delta^8\left(\eta^s + \frac{[as]}{[ab]}\eta^b + \frac{[sb]}{[ab]}\eta^a\right)\left(\frac{1}{[is]}h^\alpha_s\frac{\partial}{\partial h^\alpha_i} + \frac{1}{[sa]}h^\alpha_s\frac{\partial}{\partial h^\alpha_a}\right)\right] \\
&= \sum_{i=1}^{n}\langle si\rangle[is]\left[\frac{\langle ia\rangle}{\langle is\rangle^2\langle sa\rangle}\left(\tilde{h}^{\dot\alpha}_s\frac{\partial}{\partial\tilde{h}^{\dot\alpha}_i} + \eta^s_A\frac{\partial}{\partial\eta^i_A}\right) + \frac{[ia]}{[is]^2[sa]}\delta^8\left(\eta^s + \frac{[as]}{[ab]}\eta^b + \frac{[sb]}{[ab]}\eta^a\right)\left(h^\alpha_s\frac{\partial}{\partial h^\alpha_i}\right)\right]
\end{aligned}$$

where we used the momentum conservation

$$\sum_i\langle si\rangle[ia] = \sum_i[si]\langle ia\rangle = 0.$$

Note that in the soft superfield, $\eta_s \to 0$ gives the positive helicity soft graviton and $\delta^8(\eta_s)$ gives the negative helicity soft graviton. Thus we only retain these terms in the soft factor. Thus we get

$$\text{Soft}^{\text{SG}}_{\text{subleading}}(a,s,b) = \sum_{i=1}^{n}\frac{[is]\langle ia\rangle}{\langle si\rangle\langle sa\rangle}\tilde{h}^{\dot\alpha}_s\frac{\partial}{\partial\tilde{h}^{\dot\alpha}_i} + \frac{\langle si\rangle[ia]}{[is][sa]}\delta^8(\eta^s)h^\alpha_s\frac{\partial}{\partial h^\alpha_i}$$

which is the sum of soft factors for positive and negative helicity soft graviton in pure gravity [113, Eq. 2.9]. Note that in the above formula, the momenta p_a acts as a reference vector and hence can be taken to be any null vector r. This is an indication of the diffeomorphism symmetry of gravity amplitudes. We can thus rewrite the soft factor as

$$\text{Soft}^{\text{SG}}_{\text{subleading}}(a,s,b) = \sum_{i=1}^{n}\frac{[is]\langle ir\rangle}{\langle si\rangle\langle sr\rangle}\tilde{h}^{\dot\alpha}_s\frac{\partial}{\partial\tilde{h}^{\dot\alpha}_i} + \frac{\langle si\rangle[ir]}{[is][sr]}\delta^8(\eta^s)h^\alpha_s\frac{\partial}{\partial h^\alpha_i} \tag{4.39}$$

4.5.2 Leading soft gravitino limit

To calculate the soft limit of gravitinos, we use the results of [214]. Under the holomorphic soft limit of the superfield, we have

$$\mathcal{M}_{n+1}(\Psi_s,\Psi_1,\ldots,\Psi_n)=\left(\frac{1}{\varepsilon^3}\mathcal{S}^{(0)}+\frac{1}{\varepsilon^2}\mathcal{S}^{(1)}+\frac{1}{\varepsilon}\mathcal{S}^{(2)}\right)\mathcal{M}_n(\Psi_1,\ldots,\Psi_n)+O\left(\varepsilon^0\right). \quad (4.40)$$

The leading soft factor[2] is same with the one in pure gravity:

$$\mathcal{S}^{(0)}=\sum_{i=1}^{n}\frac{[si]\langle ri\rangle^2}{\langle si\rangle\langle rs\rangle^2}=S^{(0)}. \quad (4.41)$$

The sub-leading soft operator is given by

$$\mathcal{S}^{(1)}=\sum_{i=1}^{n}\frac{[si]\langle ri\rangle}{\langle si\rangle\langle rs\rangle}\left(\tilde{\lambda}_{s\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_{i\dot{\alpha}}}+\eta_{sA}\frac{\partial}{\partial\eta_{iA}}\right)=S^{(1)}+\eta_{sA}\mathcal{S}^{A(1)}. \quad (4.42)$$

where

$$\mathcal{S}^{A(1)}=\sum_{i=1}^{n}\frac{[si]\langle ri\rangle}{\langle si\rangle\langle rs\rangle}\frac{\partial}{\partial\eta_{iA}}$$

Here, the leading soft gravitino operator involves the first order derivatives with respect to the Grassmannian variables η_i's. These term will preserves the total helicity as well as SU(8) R-symmetry.

The sub-sub-leading soft factor is given by

$$\mathcal{S}^{(2)}=S^{(2)}+\eta_{sA}\mathcal{S}^{A(2)}+\frac{1}{2}\eta_{sA}\eta_{sB}\mathcal{S}^{AB(2)} \quad (4.43)$$

where

$$\begin{aligned} S^{(2)}&=\frac{1}{2}\sum_{i=1}^{n}\frac{[si]}{\langle si\rangle}\tilde{\lambda}_{s\dot{\alpha}}\tilde{\lambda}_{s\dot{\beta}}\frac{\partial^2}{\partial\tilde{\lambda}_{i\dot{\alpha}}\partial\tilde{\lambda}_{i\dot{\beta}}},\\ \mathcal{S}^{A(2)}&=\sum_{i=1}^{n}\frac{[si]}{\langle si\rangle}\tilde{\lambda}_{s\dot{\alpha}}\frac{\partial^2}{\partial\tilde{\lambda}_{i\dot{\alpha}}\partial\eta_{aA}},\\ \mathcal{S}^{AB(2)}&=\sum_{i=1}^{n}\frac{[si]}{\langle si\rangle}\frac{\partial^2}{\partial\eta_{aB}\partial\eta_{aA}}.\end{aligned} \quad (4.44)$$

[2] note that we have made explicit the reference vector r which was taken to be p_n in [214]

We now expand the generic superamplitude on the left-hand side of (4.40) in the Grassmann odd variable η_s of the soft superfield:

$$\begin{aligned}\mathcal{M}_{n+1}(\Psi_s,\Psi_1,\ldots,\Psi_n) =& \mathcal{M}_{n+1}(H_{s+},\Psi_1,\ldots,\Psi_n)+\eta_{sA}\mathcal{M}_{n+1}\left(S^A_{s+},\Psi_1,\ldots,\Psi_n\right)\\ &+\frac{1}{2}\eta_{sA}\eta_{sB}\mathcal{M}_{n+1}\left(G^{AB}_{s+},\Psi_1,\ldots,\Psi_n\right)+\cdots\end{aligned} \tag{4.45}$$

and compare with the right-hand side of (4.40) to get the following soft limits:

Soft Superfields	**Superamplitude expansion on $\varepsilon \to 0$**
Soft graviton	$\mathcal{M}_{n+1}(H_{s+},\ldots) = \left(\frac{1}{\varepsilon^3}S^{(0)}+\frac{1}{\varepsilon^2}S^{(1)}+\frac{1}{\varepsilon}S^{(2)}\right)\mathcal{M}_n + O\left(\varepsilon^0\right)$
Soft gravitino	$\mathcal{M}_{n+1}(S^A_{s+},\ldots) = \left(\frac{1}{\varepsilon^2}\mathcal{S}^{A(1)}+\frac{1}{\varepsilon}\mathcal{S}^{A(2)}\right)\mathcal{M}_n + O\left(\varepsilon^0\right)$
Soft graviphoton	$\mathcal{M}_{n+1}(G^{AB}_{s+}\ldots) = \frac{1}{\varepsilon}\mathcal{S}^{AB(2)}\mathcal{M}_n + \mathcal{O}\left(\varepsilon^0\right)$
Soft graviphotino	$\mathcal{M}_{n+1}(\chi^{ABC}_s\ldots) = \frac{0}{\varepsilon} + \mathcal{O}\left(\varepsilon^0\right)$
Soft scalar	$\mathcal{M}_{n+1}(\Phi^{ABCD}_s,\ldots) = \frac{0}{\varepsilon} + \mathcal{O}\left(\varepsilon^0\right)$

Table 4.15: Various soft limit expansion of the superamplitude

One can easily check that the soft graviton limit obtained here coincides with our calculations in Subsection 4.5.1. We also see that there are no soft divergences for graviphotino and scalar.

4.6 Conclusion

In this work, we have computed the soft and collinear limits of the maximally supersymmetric $\mathcal{N}=8$ supergravity theory in four spacetime dimensions using the double copy relations in both soft and collinear sectors of $\mathcal{N}=4$ Super Yang-Mills. The computation was done in the *celestial* basis appropriate for applications to celestial holography. An important point in our application of double copy is a different choice of self-duality condition for scalars. The constraints imposed here differ in signs: $\alpha_4=\tilde{\alpha}_4=-1$ and $\alpha_8=1$. This choice is motivated by our desire to combine the collinear limits for different factorizations of $\mathcal{N}=4$ SYM to $\mathcal{N}=8$ supergravity. Based on the factorization of states in the gravity theory in terms of states in the gauge theory, we are able to constrain and determine the R-symmetry indices in the collinear limit. This is also the novelty of this work.

In the next chapter, we will see the asymptotic symmetry algebra construction of maximally supersymmetric $\mathcal{N} = 8$ supergravity using the results of this chapter, but in the context of flat space holography.

CHAPTER 5

ASYMPTOTIC SYMMETRY ALGEBRA OF $\mathcal{N} = 8$ SUPERGRAVITY

In this chapter, the work done was conducted in collaboration with Nabamita Banerjee and Ranveer K. Singh, Phys. Rev. D 109, 046010, published on 26th February 2024.

5.1 Introduction

Here, we calculate the asymptotic symmetries of the four-dimensional maximally supersymmetric $\mathcal{N} = 8$ supergravity using the CCFT prescription. In celestial CFT of supergravity, the stress tensor is generated by the shadow transform [351, 388, 389] of the soft graviton operator suitably modified to obtain the correct OPE [187][1], while the supercurrent is generated by the soft gravitino operator [7].

For $\mathcal{N} > 1$, the global symmetry algebra contains an additional R-symmetry, and hence naïvely one would expect that the asymptotic algebra would contain an infinite dimensional extension of the global R-symmetry algebra as well. It was shown in [390] that for $\mathcal{N} = 2$, even for the $\mathrm{U}(1)^{\mathcal{N}}$ subgroup of the R-symmetry group $\mathrm{U}(\mathcal{N})$ which only scales the supercharges, such an infinite dimensional extension is mathematically inconsistent. For the present chapter, we study the celestial amplitudes of $\mathcal{N} = 8$ supergravity and use the soft and collinear limits calculated in chapter 4 to compute the Ward identities and the OPE of conformal operators in the corresponding CCFT.

In chapter 4, we computed the soft and collinear amplitudes of the $\mathcal{N} = 8$ supergravity from $\mathcal{N} = 4$ Super Yang-Mills Theory. This double copy is not established at the level of amplitude. However, one can interpret this as the *double copy of the universalities* in soft and collinear sectors. Although there has been recent work in the literature towards the double copy of the complete *celestial* amplitudes [391], we are focused on implementing the *celestial* map defined in Eq.(2.13) in chapter 2 on supergravity amplitudes to construct the *celestial* superamplitude after the successful use of double copy from the previous chapter in this work.

We construct the stress tensor and the supercurrents of the theory using the shadow transforms of soft graviton and soft gravitino operators. Since the scalars and graviphotinos do not have soft divergences (see Table 4.15, chapter 4 [392]), we are only left with soft graviphoton operators. The R-symmetry current (if any) can then only be constructed using the soft

[1] see Section 5.3 for more details

graviphoton operators. We construct the most general such operator present in the CCFT and show that the operator is trivial by requiring that the modes of this operator extend the $\mathrm{SU}(8)_R$ R-symmetry algebra.

The chapter is organized as follows: in Section 5.2, we set up our notations and record some definitions and results about the soft and collinear limits in the CCFT of $\mathcal{N} = 8$ supergravity used later in the chapter. In Section 5.3, we construct the symmetry currents and compute their OPEs. We also construct the possible R-symmetry currents and show that the requirements of R-symmetry extension make the current trivial. Finally, in Section 5.4, we list the full $\mathcal{N} = 8$ $\mathfrak{sbms}_4$ algebra. We conclude in Section 5.5 by summarising our results and emphasizing our future goals of the study. The Appendices C contain the OPEs of various conformal operators in the Mellin basis computed from the results in chapter 4 and a detailed calculation of the OPE of the possible R-symmetry currents.

5.2 Notations and preliminaries

In this section, we set up the notations for celestial amplitudes and soft and collinear limits in supergravity.

5.2.1 OPEs of celestial operators in $\mathcal{N} = 8$ Supergravity

Let $\{\eta_A\}_{A=1}^8$ be the Grassmann coordinates on the $\mathcal{N} = 8$ superspace. We can package the on-shell degrees of freedom in $\mathcal{N} = 8$ supergravity in an on-shell superfield defined as

$$\begin{aligned}\Psi(p,\eta) = H^+(p) + \eta_A \psi_+^A(p) + \eta_{AB} G_+^{AB}(p) + \eta_{ABC} \chi_+^{ABC}(p) \\ + \eta_{ABCD} \Phi^{ABCD}(p) + \tilde{\eta}^{ABC} \chi_{ABC}^-(p) + \tilde{\eta}^{AB} G_{AB}^-(p) + \tilde{\eta}^A \psi_A^-(p) + \tilde{\eta} H^-(p)\end{aligned} \tag{5.1}$$

where we have introduced the notation

$$\begin{aligned}\eta_{A_1 \ldots A_n} &\equiv \frac{1}{n!} \eta_{A_1} \cdots \eta_{A_2} \\ \tilde{\eta}^{A_1 \ldots A_n} &\equiv \varepsilon^{A_1 \ldots A_n B_1 \ldots B_{8-n}} \eta_{B^1 \ldots B^{8-n}} \\ \tilde{\eta} &\equiv \prod_{A=1}^{8} \eta^A.\end{aligned} \tag{5.2}$$

The fields $H^\pm$ represent positive and negative helicity graviton, G_+^{AB} and G_{AB}^- represent positive and negative helicity graviphotons, ψ_+^A and ψ_A^- represent positive and negative helicity gravitinos, χ_+^{ABC} and χ_{ABC}^- represent positive and negative helicity graviphotinos and finally Φ^{ABCD}

represent the real scalars. The *superamplitude* is then defined by

$$\mathcal{M}_n(\{p_1,\eta^1\},\ldots\{p_n,\eta^n\}) = \langle \Psi_1(p_1,\eta^1)\ldots\Psi_n(p_n,\eta^n)\rangle. \tag{5.3}$$

This superfield can be Mellin transformed in the usual way to obtain a *celestial superfield* on $\mathcal{CS}^2$, but it turns out that the component fields will have the same conformal dimension [221]. This is not appropriate to work with since we want the component fields to have conformal dimensions according to their spin. Thus we work with the so-called *quasi-on-shell superfield* [221] defined as

$$\begin{aligned}\Psi_\Delta(z,\bar{z},\eta) = {} & H^+_\Delta(z,\bar{z}) + \eta_A \psi^A_\Delta(z,\bar{z}) + \eta_{AB} G^{AB}_\Delta(z,\bar{z}) + \eta_{ABC}\chi^{ABC}_\Delta(z,\bar{z}) \\ & + \eta_{ABCD}\Phi^{ABCD}_\Delta(z,\bar{z}) + \tilde{\eta}^{ABC}\bar{\chi}_{ABC\,\Delta}(z,\bar{z}) + \tilde{\eta}^{AB}\bar{G}_{AB\,\Delta}(z,\bar{z}) \\ & + \tilde{\eta}^A \bar{\psi}_{A\,\Delta}(z,\bar{z}) + \tilde{\eta} H^-_\Delta(z,\bar{z}),\end{aligned} \tag{5.4}$$

where the components are the Mellin transforms of the components fields of $\Psi(p,n)$, all with scaling dimension Δ as defined in (2.11). The celestial correlator for the component fields can then be defined as in (2.13). Using the collinear limit of the bulk amplitude, the OPEs of the celestial operators can be computed. To do this computation, we use the collinear limits computed in [392]. As an example, we calculate the OPE of two graviton operators. The celestial correlator is given by,

$$\begin{aligned}\langle \mathcal{O}_{\Delta_1,+2}\mathcal{O}_{\Delta_2,+2}\ldots\mathcal{O}_{\Delta_n,\ell_n}\rangle &= \left(\prod_{j=1}^n \int_0^\infty d\omega_j \omega_j^{\Delta_j-1}\right)\delta^4\left(\sum_i \omega_i q_i\right) M_n\left(1^{+2},2^{+2},\ldots,n\right) \\ &= \left(\prod_{j=3}^n \int_0^\infty d\omega_j \omega_j^{\Delta_j-1}\int_0^\infty d\omega_1 \int_0^\infty d\omega_2 \omega_1^{\Delta_1-1}\omega_2^{\Delta_2-1}\right) \\ &\quad \times \delta^4\left(\sum_{i=3}^n \omega_i q_i + \omega_p q_p\right)\frac{\omega_p^2}{\omega_1\omega_2}\frac{\bar{z}_{12}}{z_{12}} M_{n-1}\left(p^{+2},\ldots,n\right)\end{aligned} \tag{5.5}$$

where M_n is the bulk amplitude of component fields and we used the collinear limit

$$M_n\left(1^{+2},2^{+2},\ldots,n\right) = \frac{\omega_p^2}{\omega_1\omega_2}\frac{\bar{z}_{12}}{z_{12}} M_{n-1}\left(p^{+2},\ldots,n\right). \tag{5.6}$$

Here $p_i = \omega_i q_i,\ i=1,2$, the momenta along the collinear channel is $p = p_1 + p_2 = \omega_p q_p$ with $\omega_p = \omega_1 + \omega_2$. Now we use the following integral [221]:

$$\int_0^\infty d\omega_2\, \omega_2^{\Delta_2-1} \int_0^\infty d\omega_1\, \omega_1^{\Delta_1-1}\, \omega_1^{\alpha}\, \omega_2^{\beta}\, \omega_p^{\gamma}\, f(\omega_p) = B(\Delta_1+\alpha, \Delta_2+\beta) \int_0^\infty d\omega_p\, \omega_p^{\Delta_p-1} f(\omega_p) \tag{5.7}$$

where $\omega_p = \omega_1 + \omega_2$ and $\Delta_p = \Delta_1 + \Delta_2 + \alpha + \beta + \gamma$ and

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{5.8}$$

is the Euler beta function. We get

$$\begin{aligned} \langle \mathcal{O}_{\Delta_1,+2}\mathcal{O}_{\Delta_2,+2}\dots\mathcal{O}_{\Delta_n,\ell_n}\rangle &= \frac{\bar{z}_{12}}{z_{12}} B(\Delta_1-1,\Delta_2-1) \left(\prod_{j=3}^{n}\int_0^\infty d\omega_j \omega_j^{\Delta_j-1}\int_0^\infty d\omega_p \omega_p^{\Delta_1+\Delta_2-1}\right) \\ &\times \delta^4\left(\sum_{i=3}^{n}\omega_i q_i + \omega_p q_p\right) M_{n-1}\left(p^{+2},3,\dots,n\right) \\ &= \frac{\bar{z}_{12}}{z_{12}} B(\Delta_1-1,\Delta_2-1)\ \langle \mathcal{O}_{\Delta_1+\Delta_2,+2}\mathcal{O}_{\Delta_3,\ell_3}\dots\mathcal{O}_{\Delta_n,\ell_n}\rangle \end{aligned} \tag{5.9}$$

This gives the OPE corresponding to the two positive helicity graviton operators,

$$\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,+2}(z_2,\bar{z}_2) \sim \frac{\bar{z}_{12}}{z_{12}} B(\Delta_1-1,\Delta_2-1)\,\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2) \tag{5.10}$$

Similarly, for negative helicity gluon, we have the collinear amplitude,

$$M_n\left(1^{-2},2^{-2},\dots\right) = \frac{\omega_p^2}{\omega_1\omega_2}\frac{z_{12}}{\bar{z}_{12}} M_{n-1}\left(p^{-2},\dots,n\right) \tag{5.11}$$

Hence the OPE

$$\mathcal{O}_{\Delta_1,-2}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,-2}(z_2,\bar{z}_2) \sim \frac{z_{12}}{\bar{z}_{12}} B(\Delta_1-1,\Delta_2-1)\,\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2) \tag{5.12}$$

The collinear limit of two opposite helicity gravitons is

$$\begin{aligned} M_n\left(1^{+2},2^{-2},3,\ldots,n\right) =& \frac{\omega_1^3}{\omega_p^2\omega_2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{-2},3,\ldots,n\right) \\ &+\frac{\omega_2^3}{\omega_p^2\omega_1}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{+2},3,\ldots,n\right) \end{aligned} \tag{5.13}$$

which gives us the OPE

$$\begin{aligned} \mathcal{O}_{\Delta_1,+2}\left(z_1,z_1\right)\mathcal{O}_{\Delta_2,-2}\left(z_2,\bar{z}_2\right) = B\left(\Delta_1+3,\Delta_2-1\right)\frac{\bar{z}_{12}}{z_{12}}\mathcal{O}_{\Delta_1+\Delta_2,-2}\left(z_2,\bar{z}_2\right) \\ +B\left(\Delta_1-1,\Delta_2+3\right)\frac{z_{12}}{\bar{z}_{12}}\mathcal{O}_{\Delta_1+\Delta_2,+2}\left(z_2,\bar{z}_2\right) \end{aligned} \tag{5.14}$$

One can calculate the OPEs of all other component fields in a similar way using the collinear limits. The results are listed in Appendix C.1.

5.2.2 Soft operators in $\mathcal{N}=8$ supergravity

In the last section, we discussed the collinear limits of amplitudes. In this section, we are looking at their soft limits. As we know, soft momentum $p\to 0$ can be written as $\omega_p\to 0$ on the celestial sphere, and hence an amplitude written in the celestial coordinates can be analyzed in the soft limit of any of the external momenta. The result is a soft theorem that expresses an n-point amplitude with soft external momentum p in terms of an $(n-1)$-point amplitude along with a soft factor given by powers of ω_p^{-1}. The various powers of ω_p^{-1} then correspond to leading, sub-leading, sub-sub-leading soft theorems, and so on. Let us first define the *celestial superamplitude* as the Mellin transform of superamplitude:

$$\begin{aligned} \left\langle\prod_{n=1}^{N}\mathcal{O}_{\Delta_n}\left(z_n,\bar{z}_n,\eta^n\right)\right\rangle \equiv \left(\prod_{n=1}^{N}\int d\omega_n\omega_n^{\Delta_n-1}\right)\delta^{(4)}\left(\sum_{n=1}^{N}\omega_n q_n\right)\times \\ \mathcal{M}_N\left(\{\omega_1,z_1,\bar{z}_1,\eta^1\},\ldots\{\omega_N,z_N,\bar{z}_N,\eta^N\}\right), \end{aligned} \tag{5.15}$$

where $\mathcal{M}_N\left(\{\omega_1,z_1,\bar{z}_1,\eta^1\},\ldots\{\omega_N,z_N,\bar{z}_N,\eta^N\}\right)$ is the superampltude (5.3) written in the celestial basis. We also denote it simply by $\mathcal{M}_N\left(1,2,\ldots,N\right)$. The above expression is identical to that of (2.13), with the explicit incorporation of the Grassmann factors in the scattering amplitudes.

One can now expand both sides of (5.15) in the Grassmann parameter η_i and compare the coefficients to get the celestial amplitude of various component fields. This has been in Appendix C.2 to calculate the celestial correlator with soft graviton and soft gravitino. The celestial correlator of the leading soft graviton operator is given by

$$\left\langle J_1(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)} \frac{(\xi - z_i)^2}{(\xi - z)^2} \langle \mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1), \\ \cdots \mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i), \cdots, \mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N) \rangle \tag{5.16}$$

and

$$\left\langle \bar{J}_1(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)} \frac{(\bar{\xi} - \bar{z}_i)^2}{(\bar{\xi} - \bar{z})^2} \langle \mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1), \\ \cdots \mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i), \cdots, \mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N) \rangle \tag{5.17}$$

where

$$J_1(z,\bar{z}) = \lim_{\Delta \to 1} (\Delta - 1) \mathcal{O}_{\Delta,+2}(z,\bar{z}), \quad \bar{J}_1(z,\bar{z}) = \lim_{\Delta \to 1} (\Delta - 1) \mathcal{O}_{\Delta,-2}(z,\bar{z}) \tag{5.18}$$

are the $\Delta = 1$ soft graviton operators and $\xi \in \mathcal{CS}^2$ is a reference point. The celestial correlator of the subleading soft graviton operator is

$$\left\langle J_0(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)} \frac{(\xi - z_i)}{(\xi - z)} ((\bar{z}-\bar{z}_i)\partial_{\bar{z}_i} - 2\bar{h}_i) \\ \times \left\langle \cdots \mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i) \cdots \right\rangle \tag{5.19}$$

and

$$\left\langle \bar{J}_0(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)} \frac{(\bar{\xi} - \bar{z}_i)}{(\bar{\xi} - \bar{z})} ((z-z_i)\partial_{z_i} - 2h_i) \\ \times \left\langle \cdots \mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i) \cdots \right\rangle \tag{5.20}$$

where

$$J_0(z,\bar{z}) = \lim_{\Delta \to 0} \Delta \mathcal{O}_{\Delta,+2}(z,\bar{z}), \quad \bar{J}_0(z,\bar{z}) = \lim_{\Delta \to 0} \Delta \mathcal{O}_{\Delta,-2}(z,\bar{z}) \tag{5.21}$$

are the $\Delta = 0$ soft graviton operators and $h_i = \frac{\Delta_i + \ell_i}{2}, \bar{h}_i = \frac{\Delta_i - \ell_i}{2}$ are the conformal weights of the operator $\mathcal{O}_{\Delta_i,\ell_i}(z,\bar{z})$.

The celestial correlator of the soft gravitino operator is given by (c.f. [7, (6.2), (6.3)])

$$\left\langle J^{A}_{1/2}(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} f(A,\ell_i,*_i,*_i')(-1)^{\sigma_i} \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)} \frac{(\xi - z_i)}{(\xi - z)} \langle \cdots \mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell_i^c}(z_i,\bar{z}_i), \cdots \rangle \tag{5.22}$$

and

$$\left\langle \bar{J}_{1/2\,A}(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n^c}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \bar{f}(A,\ell_i^c,*_i,*_i')(-1)^{\sigma_i} \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)} \frac{(\bar{\xi} - \bar{z}_i)}{(\bar{\xi} - \bar{z})} \langle \cdots \mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell_i}(z_i,\bar{z}_i), \cdots \rangle \tag{5.23}$$

where

$$J^{A}_{1/2}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}} \left(\Delta - \frac{1}{2}\right) \mathcal{O}^{A}_{\Delta,+\frac{3}{2}}(z,\bar{z}), \quad \bar{J}_{1/2\,A}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}} \left(\Delta - \frac{1}{2}\right) \mathcal{O}_{\Delta,-\frac{3}{2},A}(z,\bar{z}) \tag{5.24}$$

are soft gravitino operators. Here, the superscripts $*_i$ indicate the R-symmetry index of the operator. We have put the R-symmetry index $*_i$ as a superscript for brevity but it can also be on subscript depending on the helicity of the operator. Here the number of fermions preceding particle i, $\sigma_i = 1$ if $\ell_i \in \mathbb{Z} + \frac{1}{2}$ and 0 otherwise (see [7] for detailed explanation).

As explained in Appendix C.2, the positive helicity soft gravitino operator only acts on celestial operators $\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)$ with

$$\ell_i \in \{-3/2, -1, -1/2, 0, +1/2, +1, +3/2, +2\}. \tag{5.25}$$

while the negative helicity soft gravitino operator acts on celestial operators $\mathcal{O}^{*_i}_{\Delta_i,\ell_i^c}(z_i,\bar{z}_i)$ with

$$\ell_i^c \in \{-2, -3/2, -1, -1/2, 0, +1/2, +1, +3/2\}. \tag{5.26}$$

The factors $f(A,\ell_i,*_i,*_i'), \bar{f}(A,\ell_i^c,*_i,*_i')$ are the R-symmetry factors that we can determine using the collinear limits given above. From (5.22) and (5.23) it is clear that the first argument of f is the R-symmetry index of the soft gravitino operator itself, the second and third arguments are the helicity ℓ_i and R-symmetry index $*_i$ respectively of the operator $\mathcal{O}^{*_i}_{\Delta_i,\ell_i}$ which the soft gravitino will act on. Lastly, the fourth argument will be the R-symmetry index $*_i'$ of the resultant operator. Similarly, it goes for $\bar{f}$. As an example, we can see from the OPE in Eq.(C.1) that when $\ell_i = -\frac{3}{2}$, $f(A,-3/2,B,*_i') = \delta^A_B$. Since the resulting particle $\ell = -2$ has no R-symmetry index the $*_i'$ entry is empty.

The soft graviphoton limit can be calculated using the OPEs of the graviphoton operator with various conformal operators. These OPEs are listed in Appendix C.1. Soft limits correspond to the values of scaling dimension Δ of the graviphoton operator for which the beta functions appearing in the OPEs have poles. From Appendix C.1 we see that the OPEs of the graviphoton operator with various other operators involve[2] $B(\Delta,*)$. Since $B(\Delta,*)$ has poles at all non-positive integer values of Δ, the leading soft limit of the graviphoton operator is $\Delta \to 0$, and all other negative, integral values are subleading. In Subsection 5.3.2, we will need the leading soft graviphoton limit.

Finally, as noted in [392], graviphotino and scalars are trivial in the soft limit and hence do not correspond to any global symmetry [7]. So we do not consider them further.

5.3 Asymptotic Symmetry Generators in $\mathcal{N}=8$ SUGRA

Let us first consider the obvious global symmetries of $\mathcal{N}=8$ supergravity. The global symmetry algebra consists of the Poincaré algebra and the $\mathcal{N}=8$ supersymmetry algebra, together called the $\mathcal{N}=8$ super-Poincaré algebra and $\mathrm{SU}(8)_R$ $R-$symmetry algebra. At null infinity, we expect to obtain infinite dimensional extensions of these algebras. Following previous works [7, 186, 188], we can easily construct the currents that extend the super-Poincaré algebra, we call this algebra the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra. We start by constructing the currents for the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra.

5.3.1 $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra currents

The $\mathfrak{bms}_4$ part of the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra is known to be generated [186] by the shadow transform of the $\Delta=0$ graviton operator suitably modified as discussed below. This is called the generator of superrotations and the level one descendant of the $\Delta=1$ graviton operator is called the generator of supertranslations on the celestial sphere. Let us define the shadow

[2] The OPE of two graviphoton operators with opposite helicity involves another term, c.f. Eq. (C.2). One of the terms in the OPE vanishes depending on which of the two helicities of the graviphoton we take to be soft. See Appendix C.3 for such calculations.

transforms $T_0(z,\bar{z})$ and $\overline{T}_0(z,\bar{z})$ as:

$$\begin{aligned} T_0(z,\bar{z}) &= \lim_{\Delta\to 0}\frac{3!\Delta}{2\pi}\int d^2z'\frac{1}{(z-z')^4}\mathcal{O}_{\Delta,-2}\left(z',\bar{z}'\right) \\ \overline{T}_0(z,\bar{z}) &= \lim_{\Delta\to 0}\frac{3!\Delta}{2\pi}\int d^2z'\frac{1}{(\bar{z}-\bar{z}')^4}\mathcal{O}_{\Delta,+2}\left(z',\bar{z}'\right) \end{aligned} \tag{5.27}$$

It has been argued in [187] that the above shadow transform operator does not satisfy the usual OPE of a stress tensor. In particular, the T_0T_0 OPE has an extra term that does not vanish as shown in [187, Appendix C] unless we modify the stress tensor appropriately. The origin of the problem is the observation that $T_0(z)$ is not holomorphic:

$$\bar{\partial}T_0 = -\frac{1}{2}\partial^3\bar{J}_0(z,\bar{z}) \tag{5.28}$$

where J_0 is the $\Delta=0$ soft graviton operator defined in (5.21). Hence the modified stress tensor can be defined as follows:

$$T_{\text{mod}} := T_0 + \frac{1}{2}\partial^3\varepsilon_{\bar{J}_0} \tag{5.29}$$

where

$$\varepsilon_{\bar{J}_0} := \int_{\bar{z}_0}^{\bar{z}} d\bar{w}\bar{J}_0(z,\bar{w}) \tag{5.30}$$

with z_0 as a reference point. Then it has been shown that the modified stress tensor satisfies the correct $T_{\text{mod}}T_{\text{mod}}$ OPE [187, Appendix C]. From now on, we omit the subscript "mod" and $T,\overline{T}$ will denote the modified stress tensor. Using the soft limits (5.16), (5.17), (5.19) and (5.20) and performing the same calculations as in [186], we arrive at the OPE

$$\begin{aligned} T(z)\mathcal{O}_{\Delta,\ell}(w,\bar{w}) &= \frac{h}{(z-w)^2}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \frac{1}{z-w}\partial_w\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \text{ regular.} \\ \overline{T}(\bar{z})\mathcal{O}_{\Delta,\ell}(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\mathcal{O}_{\Delta,\ell}(w,\bar{w}) + \text{regular.} \end{aligned} \tag{5.31}$$

The supertranslations generator $P(z)$, $\overline{P}(z)$ are defined as:

$$\begin{aligned} P(z) &= \lim_{\Delta\to 1}\frac{(\Delta-1)}{4}\partial_{\bar{z}}\mathcal{O}_{\Delta,+2}(z,\bar{z}) \\ \overline{P}(\bar{z}) &= \lim_{\Delta\to 1}\frac{(\Delta-1)}{4}\partial_z\mathcal{O}_{\Delta,-2}(z,\bar{z}). \end{aligned} \tag{5.32}$$

For $P(z)$ we have

$$P(z)\mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \frac{1}{z-w}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{ regular} \tag{5.33}$$

and similar OPEs hold for $\overline{P}(\bar{z})$ with conjugated poles. These operators are related to the supertranslation generator $\mathcal{P}(z,\bar{z})$, which is a primary field operator of conformal weight $(\frac{3}{2},\frac{3}{2})$. By contour integrals [186]:

$$P(z) = \frac{1}{2\pi i}\oint d\bar{z}\mathcal{P}(z,\bar{z}), \quad \overline{P}(\bar{z}) = \frac{1}{2\pi i}\oint dz\mathcal{P}(z,\bar{z}). \tag{5.34}$$

The supertranslation satisfies the OPE

$$\mathcal{P}(z,\bar{z})\mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \frac{1}{z-w}\frac{1}{\bar{z}-\bar{w}}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{regular}. \tag{5.35}$$

The supercurrent for $\mathcal{N}=1$ supersymmetry was constructed in [7]. We will see that the same construction will give us the 8 supercurrents for $\mathcal{N}=8$ supersymmetry. We thus define the supercurrents as the shadow transform of the $\Delta=\frac{1}{2}$ gravitino operator:

$$\begin{aligned} S_A(z) &= \lim_{\Delta\to\frac{1}{2}}\frac{\Delta-\frac{1}{2}}{\pi}\int d^2z'\frac{1}{(z-z')^3}\mathcal{O}_{A;\Delta,-\frac{3}{2}}(z',\bar{z}') \\ \overline{S}^A(\bar{z}) &= \lim_{\Delta\to\frac{1}{2}}\frac{\Delta-\frac{1}{2}}{\pi}\int d^2z'\frac{1}{(\bar{z}-\bar{z}')^3}\mathcal{O}^A_{\Delta,+\frac{3}{2}}(z',\bar{z}') \end{aligned} \tag{5.36}$$

Note that the above operators are also not holomorphic since

$$\bar{\partial}S_A(z,\bar{z}) = \lim_{\Delta\to 1/2}(\Delta-\frac{1}{2})\partial_z^2\mathcal{O}_{A;\Delta,-\frac{3}{2}}(z,\bar{z}) = \partial^2\bar{J}_{1/2\,A}(z,\bar{z}) \neq 0, \tag{5.37}$$

where $\bar{J}_{1/2\,A}(z,\bar{z})$ is the leading soft gravitino operator defined in (5.24). One can modify it in a similar way as in Eq. (5.29). Put

$$\varepsilon_{\bar{J}_{1/2}\,A}(z,\bar{z}) := \int_{\bar{z}_0}^{z} d\bar{w}\bar{J}_{1/2\,A}(w,\bar{w}) \tag{5.38}$$

where z_0 is a reference point and define

$$S^A_{\text{mod}} := S^A - \partial^2\varepsilon^A_{\bar{J}_{1/2}}. \tag{5.39}$$

We emphasize that this modification is not required at the quantum level since the OPEs of S^A are as expected for a supercurrent. So we continue to use the shadow transform of the leading soft gravitino operator as the supercurrent without any modification.

Following the calculations of [7, Section 7], it is straightforward to see that

$$\begin{aligned} T(z)S_A(w) &= \frac{3}{2}\frac{S_A(w)}{(z-w)^2}+\frac{\partial S_A(w)}{z-w}+\text{ regular}, \\ \overline{T}(\bar{z})\overline{S}^A(\bar{w}) &= \frac{3}{2}\frac{\overline{S}^A(\bar{z})}{(\bar{z}-\bar{w})^2}+\frac{\bar{\partial}\overline{S}^A(\bar{w})}{\bar{z}-\bar{w}}+\text{ regular}. \end{aligned} \tag{5.40}$$

and the OPEs $T\overline{S}^A$ and $\overline{T}S_A$ are regular. These OPEs confirm the conformal weights of S_A and $\overline{S}_A$ as $(\frac{3}{2},0)$ and $(0,\frac{3}{2})$ respectively. We now want to show that

$$:\{S_B(z),\overline{S}^A(\bar{z})\}: \;=\; :S_B(z)\overline{S}^A(\bar{z})+\overline{S}^A(\bar{z})S_B(z): \;=\delta^A_B\mathcal{P}(z,\bar{z}). \tag{5.41}$$

Using the gravitino soft limit (5.22) and (5.23) and the leading graviton limits (5.16) and (5.17) and following the calculations in [7, Section 7.3], we get[3]

$$\begin{aligned} &\left\langle S_B(z)\overline{S}^A(\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &=\delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(\bar{w}-\bar{z})^2}\frac{\bar{z}-\bar{z}_i}{z-z_i}+\frac{1}{\bar{z}-\bar{w}}\frac{1}{z-z_i}+\frac{1}{\bar{w}-\bar{z}_i}\frac{1}{z-z_i}\right]\left\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle \\ &-\sum_{i=3}^{N}f(A,\ell_i,*_i,*_i')\bar{f}(B,\ell_i-1/2,*_i',*_i'')\frac{1}{z-z_i}\frac{1}{\bar{w}-\bar{z}_i}\left\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle \\ &+\sum_{\substack{i,j=3\\ i\neq j}}^{N}(-1)^{\sigma_i+\sigma_j}f(A,\ell_i,*_i,*_i')\bar{f}(B,\ell_j,*_j,*_j')\frac{1}{z-z_i}\frac{1}{\bar{w}-\bar{z}_j}\left\langle\cdots\mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell_i-\frac{1}{2}}(z_i,\bar{z}_i),\cdots,\right. \\ &\qquad\qquad\left.\mathcal{O}^{*_j'}_{\Delta_j+\frac{1}{2},\ell_j+\frac{1}{2}}(z_j,\bar{z}_j)\cdots\right\rangle \end{aligned} \tag{5.42}$$

where the factors $f(A,\ell_i,*_i,*_i'),\bar{f}(B,\ell_j,*_j,*_j')$ are the R-symmetry factors that appear on taking the soft or collinear limit depending on the spins and helicities of the soft and collinear particles. In this notation, the first argument of f is the R-symmetry index of the positive helicity soft

[3] note that we do not separate the operators in the correlator according to their spins ℓ,ℓ^c unlike [7] since there is an overlap in the ranges of the two spins. So, the spins are assumed to be arbitrary in the correlators in this calculation.

gravitino, second argument is the spin (and helicity) of one of the hard[4] particles, the third argument is the R-symmetry index of that hard particle (left implicit for generality) and the fourth argument is the resulting R-symmetry index of the hard particle after the soft limit is taken (again left implicit for generality). The notation for $\bar{f}$ is similar. It is understood that if the spins do not belong to the required range specified in (C.29) and (C.30) then $f,\bar{f}=0$. Similarly

$$\begin{aligned}
&\left\langle \overline{S}^A(\bar{z})S_B(w)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&=\delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(w-z)^2}\frac{z-z_i}{\bar{z}-\bar{z}_i}+\frac{1}{z-w}\frac{1}{\bar{z}-\bar{z}_i}+\frac{1}{\bar{z}-\bar{z}_i}\frac{1}{w-z_i}\right]\left\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle \\
&-\sum_{i=3}^{N}\bar{f}(B,\ell_i,*_i,*_i')f(A,\ell_i+1/2,*_i',*_i'')\frac{1}{w-z_i}\frac{1}{\bar{z}-\bar{z}_i}\left\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle \\
&-\sum_{\substack{i,j=3\\ i\neq j}}^{N}(-1)^{\sigma_i+\sigma_j}\bar{f}(B,\ell_i,*_i,*_i')f(A,\ell_j,*_j,*_j')\frac{1}{w-z_i}\frac{1}{\bar{z}-\bar{z}_j}\left\langle\cdots\mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell_i+\frac{1}{2}}(z_i,\bar{z}_i),\cdots,\right. \\
&\qquad\qquad\left.\mathcal{O}^{*_j'}_{\Delta_j+\frac{1}{2},\ell_j-\frac{1}{2}}(z_j,\bar{z}_j)\cdots\right\rangle
\end{aligned} \tag{5.43}$$

Thus the anticommutator is

$$\begin{aligned}
&\left\langle\left(\overline{S}^A(\bar{z})S_B(w)+S_B(z)\overline{S}^A(\bar{w})\right)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&=\delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(w-z)^2}\frac{z-z_i}{\bar{z}-\bar{z}_i}+\frac{1}{z-w}\frac{1}{\bar{z}-\bar{z}_i}+\frac{1}{\bar{z}-\bar{z}_i}\frac{1}{w-z_i}\right]\left\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle \\
&+\delta^A_B\sum_{i=3}^{N}\left[\frac{1}{(\bar{w}-\bar{z})^2}\frac{\bar{z}-\bar{z}_i}{z-z_i}+\frac{1}{\bar{z}-\bar{w}}\frac{1}{z-z_i}+\frac{1}{\bar{w}-\bar{z}_i}\frac{1}{z-z_i}\right]\left\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle \\
&-\sum_{i=3}^{N}f(A,\ell_i,*_i,*_i')\bar{f}(B,\ell_i-1/2,*_i',*_i'')\frac{1}{z-z_i}\frac{1}{\bar{w}-\bar{z}_i}\left\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle \\
&-\sum_{i=3}^{N}\bar{f}(B,\ell_i,*_i,*_i')f(A,\ell_i+1/2,*_i',*_i'')\frac{1}{w-z_i}\frac{1}{\bar{z}-\bar{z}_i}\left\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle.
\end{aligned} \tag{5.44}$$

Here, in the last terms in Eq.(5.42) and Eq.(5.43), we have relative signs; hence, both terms cancel. One can notice that the relative sign is due to the action of S and $\overline{S}$ on different clusters for $i<j$ and $i>j$ in both terms. Then we see that the normal ordered current : $\{S_B(z),\overline{S}^A(\bar{z})\}$:

[4] that is not soft

satisfies

$$\begin{aligned}
&\left\langle :\{S_B(z),\overline{S}^A(\bar{z})\}:\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\
&=2\delta^A_B\sum_{i=3}^{N}\frac{1}{\bar{z}-\bar{z}_i}\frac{1}{z-z_i}\left\langle\cdots\mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle\\
&-\sum_{i=3}^{N}f(A,\ell_i,*_i,*_i')\bar{f}(B,\ell_i-1/2,*_i',*_i'')\frac{1}{z-z_i}\frac{1}{\bar{z}-\bar{z}_i}\left\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle\\
&-\sum_{i=3}^{N}\bar{f}(B,\ell_i,*_i,*_i')f(A,\ell_i+1/2,*_i',*_i'')\frac{1}{z-z_i}\frac{1}{\bar{z}-\bar{z}_i}\left\langle\cdots\mathcal{O}^{*_i''}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots\right\rangle.
\end{aligned}\tag{5.45}$$

We now show that for any ℓ_i, R-symmetry factors in the last two sums reduce to δ^A_B. Let us start with $\ell_i=+2$ in which case $*_i,*_i''$ is empty. Moreover, in this case, $\bar{f}(B,+2,*_i,*_i')=0$ so that we only have one term to analyse. From the OPEs in Appendix C.1, we see that $*_i'=A$ and

$$f(A,+2,\_,*_i')\bar{f}(B,+3/2,*_i',\_)\mathcal{O}_{\Delta_i+1,+2}(z_i,\bar{z}_i)=\delta^A_B\mathcal{O}_{\Delta_i+1,+2}(z_i,\bar{z}_i). \tag{5.46}$$

The case $\ell_i=+\frac{3}{2}$ is more interesting. Suppose $*_i=C$ then from the OPEs, we can easily see that $*_i'=AC$ for the second term and $*_i'$ is empty for the last term. We then have

$$f(A,+3/2,C,*_i')\bar{f}(B,+1,*_i',*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)=2!\delta^{[A}_B\mathcal{O}^{C]}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i) \tag{5.47}$$

and similarly

$$\bar{f}(B,+3/2,C,\_)f(A,+2,\_,*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)=\delta^C_B\mathcal{O}^A_{\Delta_i+1,+3/2}(z_i,\bar{z}_i) \tag{5.48}$$

We can clearly see that the sum of the last two terms is simply $\delta^A_B\mathcal{O}^C_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)$. The case $\ell_i=-\frac{3}{2}$ is similar. Let us now analyze the case $\ell_i=+1$ in which case $*_i=CD$. We get

$$\begin{aligned}
&f(A,+1,CD,*_i')\bar{f}(B,+1/2,*_i',*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+2}(z_i,\bar{z}_i)\\
&\qquad=\bar{f}(B,+1/2,*_i',ACD)\mathcal{O}^{ACD}_{\Delta_i+\frac{1}{2},+1/2}(z_i,\bar{z}_i)\\
&\qquad=3\delta^{[A}_B\mathcal{O}^{CD]}_{\Delta_i+1,+1}(z_i,\bar{z}_i).
\end{aligned}\tag{5.49}$$

Similarly

$$\bar{f}(B,+1,CD,*_i')f(A,+3/2,*_i',*_i'')\mathcal{O}^{*_i''}_{\Delta_i+1,+3/2}(z_i,\bar{z}_i)=-2!\delta^{[C}_B\mathcal{O}^{D]A}_{\Delta_i+1,+1}(z_i,\bar{z}_i) \tag{5.50}$$

which finally implies

$$\begin{aligned}3\delta_B^{[A}\mathcal{O}^{CD]} - 2!\delta_B^{[C}\mathcal{O}^{D]A} &= \frac{1}{2}\left[(\delta_B^A\mathcal{O}^{CD} - \delta_B^A\mathcal{O}^{DC}) + (\delta_B^C\mathcal{O}^{DA} - \delta_B^C\mathcal{O}^{AD}) + (\delta_B^D\mathcal{O}^{AC} - \delta_B^D\mathcal{O}^{CA})\right] \\ &\quad - \delta_B^C\mathcal{O}^{DA} + \delta_B^D\mathcal{O}^{CA} \\ &= \delta_B^A\mathcal{O}^{CD}\end{aligned} \tag{5.51}$$

The case $\ell_i = -1$ is similar. The same calculation as in $\ell_i = 1$ recurs for the cases $\ell_i = 1/2, 0$.

These calculations simplify the OPE (5.45). We get

$$\left\langle :\{S_B(z), \overline{S}^A(\bar{z})\}: \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \delta_B^A \sum_{i=3}^{N} \frac{1}{\bar{z}-\bar{z}_i}\frac{1}{z-z_i}\left\langle \cdots \mathcal{O}^{*_i}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i), \cdots\right\rangle \tag{5.52}$$

In particular,

$$:\{S_B(z), \overline{S}^A(\bar{z})\}: \mathcal{O}_{\Delta,\ell}(w,\bar{w}) = \delta_B^A \frac{1}{z-w}\frac{1}{\bar{z}-\bar{w}}\mathcal{O}_{\Delta+1,\ell}(w,\bar{w}) + \text{regular}. \tag{5.53}$$

Comparing this OPE with (5.35) readily implies the desired result

$$:\{S_B(z), \overline{S}^A(\bar{z})\}: \; = \delta_B^A \mathcal{P}(z,\bar{z}). \tag{5.54}$$

5.3.2 Possible R-symmetry current

Recall that R-symmetry acts on supercharges Q^A_α and $\overline{Q}_{\dot{\alpha}A}$, $A = 1,\ldots,8$ by multiplying a unitary matrix $U \in U(8)$. This means that the supercharges transform in the fundamental representation of the R-symmetry group. At the level of Lie algebra, we can identify the R-symmetry group as simply $\mathfrak{su}(8) \oplus \mathfrak{u}(1)$ since $\mathrm{U}(8) \cong (\mathrm{SU}(8) \times \mathrm{U}(1))/\mathbb{Z}_8$. Thus we can label the generators of R-symmetry to be T^A_B and R, where T^A_B are generators of the fundamental representation of $\mathrm{SU}(8)$ satisfying the $\mathfrak{su}(8)$ algebra:

$$\left[T^A_B, T^C_D\right] = \delta^A_D T^C_B - \delta^C_B T^A_D, \tag{5.55}$$

and R is the generator of the scaling $U(1)$. A suitable matrix representation for the generators is [215],

$$\left(T^A_B\right)^C{}_D = \delta^A_D\delta^C_B - \frac{1}{8}\delta^A_B\delta^C_D \tag{5.56}$$

T^A_B acts on the supercharges as

$$\left[T^A_B, Q^C_\alpha\right] = \left(T^A_B\right)^C{}_D Q^D_\alpha, \quad \left[T^A_B, \overline{Q}_{\dot{\alpha}C}\right] = -\left(T^A_B\right)^D{}_C \overline{Q}_{\dot{\alpha}D}. \tag{5.57}$$

We now want to construct a current $\widetilde{\mathcal{G}}^A_B(z,\bar{z})$ whose modes will extend the generators T^A_B. As will be shown in Section 5.4, the modes of the supercurrents $S_A, \overline{S}^A$ will extend the supercharges. Since the OPE of currents directly translates to the commutator of their modes within radial quantization, our currents must satisfy the OPE:

$$\begin{aligned}
\widetilde{\mathcal{G}}^A_B(z,\bar{z})S_C(w) &\sim ((z-w)\text{ singularity})\left(T^A_B\right)^D{}_C S_D(w),\\
\widetilde{\mathcal{G}}^A_B(z,\bar{z})\overline{S}^C(w) &\sim -((\bar{z}-\bar{w})\text{ singularity})\left(T^A_B\right)^C{}_D \overline{S}^D(w).
\end{aligned} \tag{5.58}$$

Note that $S_A, \overline{S}^A$ are holomorphic and antiholomorphic currents respectively, this imposes the condition that the singularities in (5.58) be holomorphic and antiholomorphic respectively. As will be shown in Section 5.4, nonholomorphic (holomorphic) singularity in the OPE of $\mathcal{G}^A_B(z,\bar{z})$ with $S_C(w)$ ($\overline{S}^C(w)$) results in nonsensical algebra. This requirement will be crucial.

The only conformal operator we are left with is the graviphoton operator. Moreover, the leading soft graviphoton operator corresponds to $\Delta = 0$, as can be inferred from the poles of the beta function in the OPEs of graviphoton operators with other operators, which are summarised in Appendix C.1. It is clear that we must consider the order independent graviphoton double soft limit with opposite helicity, that is, their normal ordered commutator (since they are bosonic). Since it contains the factor δ^{AB}_{CD}, as can be seen from the collinear limits, this can be manipulated properly to obtain the SU(8) generators. Here, we consider the most general integral transform corresponding to negative and positive helicity soft graviphotons, respectively as

$$\begin{aligned}
G_{AB}(z,\bar{z}) &= \lim_{\Delta\to 0}\frac{\Delta}{\pi}\int d^2z' \frac{1}{(z-z')^a}\frac{1}{(\bar{z}-\bar{z}')^b}\mathcal{O}_{AB;\Delta,-1}(z',\bar{z}')\\
\overline{G}^{CD}(z,\bar{z}) &= \lim_{\Delta\to 0}\frac{\Delta}{\pi}\int d^2z' \frac{1}{(\bar{z}-\bar{z}')^{a'}}\frac{1}{(z-z')^{b'}}\mathcal{O}^{CD}_{\Delta,+1}(z',\bar{z}').
\end{aligned} \tag{5.59}$$

One can easily see that we can recover the usual shadow transformation [393] by taking specific values of a and b. The operators $\mathcal{O}_{AB;0,-1}$ and $\mathcal{O}^{CD}_{0,+1}$ have conformal weights $(-\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},-\frac{1}{2})$ respectively. Hence scaling transformation reveals the conformal weights of the currents G_{AB} and $\overline{G}^{CD}$ to be $(a-\frac{3}{2},b-\frac{1}{2})$ and $(b'-\frac{1}{2},a'-\frac{3}{2})$ respectively.
Let us start with the OPE of our new currents G_{AB} and $\overline{G}^{CD}$ with any conformal primary operators.

$$\begin{aligned}&\left\langle G_{AB}(z,\bar{z})\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\lim_{\Delta_1\to 0}\frac{\Delta_1}{\pi}\int d^2z_1\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\left\langle\mathcal{O}_{AB\;\Delta_1,-1}(z_1,\bar{z}_1)\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\frac{1}{\pi}\int d^2z_1\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\left[\sum_{n=2}^{N}f(A,B,\ell_i,*_n,*_n')\frac{z_1-z_n}{\bar{z}_1-\bar{z}_n}\left\langle\cdots\mathcal{O}^{*_n'}_{\Delta_n,\ell_n+1}(z_n,\bar{z}_n)\right\rangle\right]\end{aligned}\tag{5.60}$$

where $f(A,B,\ell_i,*_n,*_n')$ contains the R-symmetry index of the operators in the correlation function, which appears on taking the collinear limit. We used the fact that $\lim_{\Delta\to 0}\Delta\,B(\Delta,*)=1$.
Now we use two basic integrals (see [394, Appendix B] for proof):

$$\begin{aligned}&\int d^2z_1\frac{1}{(z-z_1)^A}\frac{1}{(\bar{z}-\bar{z}_1)^B}\frac{(\bar{z}_1-\bar{z}_j)^s}{z_1-z_j}=C_s(A,B)\frac{1}{(z_j-z)^A\,(\bar{z}_j-\bar{z})^{B-s-1}}\\&\int d^2z_1\frac{1}{(\bar{z}-\bar{z}_1)^A}\frac{1}{(z_1-z_1)^B}\frac{(z_1-z_j)^s}{\bar{z}_1-\bar{z}_j}=C_s(A,B)\frac{1}{(\bar{z}_j-\bar{z})^A}\frac{1}{(z_j-z)^{B-s-1}}\end{aligned}\tag{5.61}$$

where

$$C_s(A,B)=\frac{(-1)^{s+A+B}(-\pi)s!}{(-B+1)(-B+2)\cdots(-B+s+1)}\tag{5.62}$$

Now performing the shadow integral for $n\neq 1$ and $s=1$,

$$\int d^2z_1\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\frac{z_1-z_n}{\bar{z}_1-\bar{z}_n}=C_1(b,a)\frac{1}{(\bar{z}_n-\bar{z})^b}\frac{1}{(z_n-z)^{a-2}}\tag{5.63}$$

We have,

$$\begin{aligned}&\left\langle G_{AB}(z,\bar{z})\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\\&=\sum_{i=2}^{N}f(A,B,\ell_i,*_i,*_i')C_1(b,a)\frac{1}{(\bar{z}_i-\bar{z})^b}\frac{1}{(z_i-z)^{a-2}}\left\langle\cdots\mathcal{O}^{*_i'}_{\Delta_i,\ell_i+1}(z_i,\bar{z}_i)\right\rangle\end{aligned}\tag{5.64}$$

Here the helicities of the conformal operators inside the correlator are restricted to $\ell_n \in \{-2,-\frac{3}{2},-1,-\frac{1}{2},0,+\frac{1}{2}$ This can be verified from the beta function singularities in the OPEs in Appendix C.1. Similarly, we can have the OPE for antiholomorphic current $\overline{G}^{CD}$ which act on the conformal primaries with helicities restricted in the range $\ell_n' \in \{-1,-\frac{1}{2},0,+\frac{1}{2},+1,+\frac{3}{2},+2\}$,

$$\begin{aligned}&\left\langle \overline{G}^{CD}(z,\bar{z})\prod_{n=2}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n'}(z_n,\bar{z}_n)\right\rangle \\ &=\sum_{n=2}^{N}\bar{f}(C,D,\ell_i',*_n,*_n')C(b',a')\frac{1}{(\bar{z}_n-\bar{z})^{b'}}\frac{1}{(z_n-z)^{a'-2}}\left\langle\cdots\mathcal{O}^{*_n'}_{\Delta_n,\ell_n'-1}(z_n,\bar{z}_n)\right\rangle\end{aligned} \tag{5.65}$$

Here, we can pair ℓ' with $\ell=\ell'+1$. Hence, we can write the OPEs as,

$$\begin{aligned}G_{AB}(z)\mathcal{O}^{*}_{\Delta,\ell}(w,\bar{w}) &\sim f(A,B,\ell,*,*')C_1(b,a)\frac{1}{(\bar{w}-\bar{z})^{b}}\frac{1}{(w-z)^{a-2}}\mathcal{O}^{*'}_{\Delta,\ell'}(z,\bar{w}) \\ \overline{G}^{CD}(z)\mathcal{O}^{*}_{\Delta,\ell'}(w,\bar{w}) &\sim \bar{f}(C,D,\ell',*,*')C_1(b',a')\frac{1}{(\bar{w}-\bar{z})^{b'}}\frac{1}{(z-z)^{a'-2}}\mathcal{O}^{*'}_{\Delta,\ell}(w,\bar{w})\end{aligned} \tag{5.66}$$

The composite current

To construct a suitable current for R-symmetry, we need to use double soft limits of the graviphoton operators. As is well known, the double soft limit of opposite helicity operators depends on the order of the soft limit. For this reason, as in [188] we consider the following operator:

$$\mathcal{G}^{CD}_{AB}(z,\bar{z};w,\bar{w}) := G_{AB}(z)\overline{G}^{CD}(\bar{w})-\overline{G}^{CD}(\bar{w})G_{AB}(z)\equiv\left[G_{AB}(z),\overline{G}^{CD}(\bar{w})\right]. \tag{5.67}$$

In order to construct a local operator, one needs to consider the normal order of this operator evaluated at $z=w,\bar{z}=\bar{z}$. We thus define

$$\begin{aligned}\mathcal{G}^{CD}_{AB}(z,\bar{z}) = :\mathcal{G}^{CD}_{AB}(z,\bar{z};z,\bar{z}): &= :G_{AB}(z)\overline{G}^{CD}(\bar{z})-\overline{G}^{CD}(\bar{z})G^{AB}(z): \\ &\equiv:\left[G_{AB}(z),\overline{G}^{CD}(\bar{z})\right]:.\end{aligned} \tag{5.68}$$

We show in Appendix C.3 that subject to the requirement of the R-symmetry current explained above (5.58), the current $\mathcal{G}_{AB}^{CD}(z,\bar{z})$ satisfies the following OPE:

$$\begin{aligned}
&\left\langle \mathcal{G}_{AB}^{CD}(z,\bar{z})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&=(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')\Bigg[f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j) \\
&\quad\times\frac{1}{(z-z_j)^{a+b'-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a'+b-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&\qquad-\bar{f}(C,D,\ell_j,*_j,*'_j)f(A,B,\ell_j-1,*'_j,*''_j) \\
&\quad\times\frac{1}{(z-z_j)^{a'+b-2}}\frac{1}{(\bar{z}-\bar{z}_j)^{a+b'-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\Bigg]
\end{aligned} \tag{5.69}$$

In particular,

$$\begin{aligned}
\mathcal{G}_{AB}^{CD}(z,\bar{z})\mathcal{O}_{E\ \Delta,-\frac{3}{2}}(w,\bar{w}) &\sim -\delta_{AB}^{CD}\frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a+b'-2}(\bar{z}-\bar{w})^{a'+b-2}}\mathcal{O}_{E\ \Delta,-\frac{3}{2}}(w,\bar{w}) \\
\mathcal{G}_{AB}^{CD}(z,\bar{z})\mathcal{O}^{E}_{\Delta,+\frac{3}{2}}(w,\bar{w}) &\sim \delta_{AB}^{CD}\frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a'+b-2}(\bar{z}-\bar{w})^{a+b'-2}}\mathcal{O}^{E}_{\Delta,+\frac{3}{2}}(w,\bar{w})
\end{aligned} \tag{5.70}$$

where we used the fact that for the gravitino operator, the R-symmetry factor in the double soft limit is $-\delta_{AB}^{CD}$. Indeed

$$\lim_{\Delta_1\to 0}\Delta_1\mathcal{O}_{AB;\Delta_1,-1}(z,\bar{z})\mathcal{O}_{E,\Delta,-\frac{3}{2}}(z_1,\bar{z}_1)=\frac{z-z_1}{\bar{z}-\bar{z}_1}\mathcal{O}_{ABE;\Delta,-\frac{1}{2}}(z_1,\bar{z}_1)$$

If $E=r$, $A=a$, $B=b$, $C=c$, $D=d$, then

$$\lim_{\Delta_2\to 0}\Delta_2\mathcal{O}^{cd}_{\Delta_2,+1}(w,\bar{w})\mathcal{O}_{abr;\Delta,-\frac{1}{2}}(z_1,\bar{z}_1)=-\delta_{ab}^{cd}\frac{\bar{w}-\bar{z}_1}{w-z_1}\mathcal{O}_{r,\Delta,-\frac{3}{2}}(z_1,\bar{z}_1)$$

In all other cases, one can check from the collinear limit in Appendix C.1 that the R-symmetry factor is $-\delta_{AB}^{CD}$. Let us construct a new current as a linear combination of our previous currents as follows:

$$\left(\widetilde{\mathcal{G}}_B^A\right)_D^C(z,\bar{z}):=-\left(\frac{1}{7}\delta_D^A\sum_{E=1}^{8}\mathcal{G}_{BE}^{EC}(z,\bar{z})+\frac{1}{56}\delta_B^A\sum_{E=1}^{8}\mathcal{G}_{ED}^{EC}(z,\bar{z})\right). \tag{5.71}$$

Using the definition of generalized Kronecker delta

$$\delta^{a_1\ldots a_n}_{b_1\ldots b_n} = \sum_{\sigma\in S_n} \text{sign}(\sigma)\delta^{a_1}_{b_{\sigma(1)}}\ldots\delta^{a_n}_{b_{\sigma(n)}}, \tag{5.72}$$

we see that

$$\sum_{E=1}^{8} \delta^{EC}_{BE} = -7\delta^{C}_{B}, \quad \sum_{E=1}^{8} \delta^{EC}_{ED} = 7\delta^{C}_{D}. \tag{5.73}$$

This gives us the OPE

$$\begin{aligned}
\left(\widetilde{\mathcal{G}}^{C}_{A}\right)^{D}_{B}(z,\bar{z})\mathcal{O}_{D\,\Delta,-\frac{3}{2}}(w,\bar{w}) &\sim \frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a+b'-2}(\bar{z}-\bar{w})^{a'+b-2}}\left(T^{C}_{A}\right)^{D}_{B}\mathcal{O}_{D\,\Delta,-\frac{3}{2}}(w,\bar{w})\\
\left(\widetilde{\mathcal{G}}^{C}_{A}\right)^{D}_{B}(z,\bar{z})\mathcal{O}^{B}_{\Delta,+\frac{3}{2}}(w,\bar{w}) &\sim -\frac{(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')}{(z-w)^{a'+b-2}(\bar{z}-\bar{w})^{a+b'-2}}\left(T^{C}_{A}\right)^{D}_{B}\mathcal{O}^{B}_{\Delta,+\frac{3}{2}}(w,\bar{w})
\end{aligned} \tag{5.74}$$

Hence $\widetilde{\mathcal{G}}^{C}_{A}$ is a candidate which can extend the R-symmetry algebra. But we see an immediate problem. The OPE of $\widetilde{\mathcal{G}}^{C}_{A}$ with supercurrents $S_D(w), \overline{S}^{B}(\bar{w})$ is given by

$$\begin{aligned}
\left(\widetilde{\mathcal{G}}^{C}_{A}\right)^{D}_{B}(z,\bar{z})S_D(w) \sim \left(T^{C}_{A}\right)^{D}_{B}\lim_{\Delta\to\frac{1}{2}}\frac{\Delta-\frac{1}{2}}{\pi}(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')\\
\times\int d^2z_1\frac{1}{(w-z_1)^3}\frac{1}{(z-z_1)^{a+b'-2}}\frac{1}{(\bar{z}-\bar{z}_1)^{a'+b-2}}\mathcal{O}_{D,\Delta,-\frac{3}{2}}(z_1,\bar{z}_1)
\end{aligned} \tag{5.75}$$

and

$$\begin{aligned}
\left(\widetilde{\mathcal{G}}^{C}_{A}\right)^{D}_{B}(z,\bar{z})\overline{S}^{B}(\bar{w}) \sim -\left(T^{C}_{A}\right)^{D}_{B}\lim_{\Delta\to\frac{1}{2}}\frac{\Delta-\frac{1}{2}}{\pi}(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')\\
\times\int d^2z_1\frac{1}{(\bar{w}-\bar{z}_1)^3}\frac{1}{(z-z_1)^{a'+b-2}}\frac{1}{(\bar{z}-\bar{z}_1)^{a+b'-2}}\mathcal{O}^{B}_{\Delta,+\frac{3}{2}}(z_1,\bar{z}_1)
\end{aligned} \tag{5.76}$$

The requirement (5.58) forces $a'+b-2=0$ in (5.75) and (5.76). But then, in view of (C.44), we get

$$a+b'-2=0 \quad \text{and} \quad a'+b-2=0 \tag{5.77}$$

and conclude that the OPE is trivial

$$\left(\widetilde{\mathcal{G}}^{C}_{A}\right)^{D}_{B}(z,\bar{z})S_D(w) \sim \text{regular}, \quad \left(\widetilde{\mathcal{G}}^{C}_{A}\right)^{D}_{B}(z,\bar{z})\overline{S}^{B}(\bar{w}) \sim \text{regular}. \tag{5.78}$$

5.4 The $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra

Let us now find out the asymptotic symmetries of the theory. The usual symmetry currents in the theory are the stress tensor $T(z), \overline{T}(\bar{z})$, which are the superrotation generators and $\mathcal{P}(z,\bar{z})$, which is the supertranslation generator. The modes of these currents generate the $\mathfrak{bms}_4$ algebra as described in [186]. As usual the generators of $\mathfrak{bms}_4$ are the modes of $T(z), \overline{T}(\bar{z})$ and $\mathcal{P}(z,\bar{z})$. Let us expand these currents in modes:

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad \overline{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}; \tag{5.79}$$

$$\mathcal{P}(z,\bar{z}) \equiv \sum_{n,m\in\mathbb{Z}} P_{n-\frac{1}{2},m-\frac{1}{2}} z^{-n-1} \bar{z}^{-m-1} \tag{5.80}$$

As discussed in [186], the modes $P_{n-\frac{1}{2},m-\frac{1}{2}}$ can be obtained from the modes of the current $P(z)$ or $\overline{P}(\bar{z})$. If we write

$$P(z) = \sum_{n\in\mathbb{Z}} P_{n-\frac{1}{2}} z^{-n-1}, \quad \overline{P}(\bar{z}) = \sum_{m\in\mathbb{Z}} \overline{P}_{m-\frac{1}{2}} z^{-m-1} \tag{5.81}$$

then

$$P_{n-\frac{1}{2},-\frac{1}{2}} = P_{n-\frac{1}{2}}, \quad P_{-\frac{1}{2},m-\frac{1}{2}} = \overline{P}_{m-\frac{1}{2}} \tag{5.82}$$

and

$$\begin{aligned} P_{n-\frac{1}{2},m-\frac{1}{2}} &= \frac{1}{i\pi(m+1)} \oint d\bar{w}\bar{w}^{m+1} \left[\bar{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}\right] \\ &= \frac{1}{i\pi(m+1)} \oint dw w^{n+1} \left[T(w), \overline{P}_{-\frac{1}{2},m-\frac{1}{2}}\right]. \end{aligned} \tag{5.83}$$

These modes satisfy the usual $\mathfrak{bms}_4$ algebra:

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} \\ &[L_n, P_{kl}] = \left(\frac{1}{2}n - k\right) P_{n+k,l}, \quad [\bar{L}_n, P_{kl}] = \left(\frac{1}{2}n - l\right) P_{k,n+l}, \end{aligned} \tag{5.84}$$

where $m,n \in \mathbb{Z}$ and $k,l \in \mathbb{Z}+\frac{1}{2}$. In addition, an infinite dimensional extension of the $\mathcal{N}=1$ supersymmetry algebra was constructed in [7]. The supercurrent was shown to be the shadow transform of the gravitino operator. In our theory, we have 8 supercurrents $S^A(z)$ and their antiholomorphic counterpart $\overline{S}_A(\bar{z})$. The OPEs (5.40) show that $S^A(z)$ and $\overline{S}_A(\bar{z})$ are conformal primaries of dimension $(\frac{3}{2},0)$ and $(0,\frac{3}{2})$ respectively. Consequently, if we expand the supercurrents as

$$
\begin{aligned}
S_A(z) &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \frac{(S_A)_k}{z^{k+\frac{3}{2}}}, \quad \text{with} \quad (S_A)_k = \frac{1}{2\pi i}\oint dz\, z^{k+\frac{1}{2}}\, S_A(z). \\
\overline{S}^A(\bar{z}) &= \sum_{l\in\mathbb{Z}+\frac{1}{2}} \frac{\overline{S}^A_l}{\bar{z}^{l+\frac{3}{2}}}, \quad \text{with} \quad \overline{S}^A_l = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{l+\frac{1}{2}}\, \overline{S}^A(\bar{z})
\end{aligned}
\tag{5.85}
$$

then we can write the commutator of these modes with the Virasoro generators as:

$$
\begin{aligned}
&[L_n,(S_A)_m] = \left(\frac{n}{2}-m\right)(S_A)_{m+n}, \quad [\bar{L}_n,(S_A)_m] = 0 \\
&\left[L_n,\overline{S}^A_m\right] = 0, \quad \left[\bar{L}_n,\overline{S}^A_m\right] = \left(\frac{n}{2}-m\right)\overline{S}^A_{m+n}.
\end{aligned}
\tag{5.86}
$$

The operator relation (5.41) gives the anticommutator

$$
\{(S_B)_m,\overline{S}^A_n\} = \delta^A_B P_{mn}, \quad m,n\in\mathbb{Z}+\frac{1}{2}. \tag{5.87}
$$

Let us now discuss the requirement of (anti)holomorphicity of the singularity in (5.58). Suppose $(\widetilde{\mathcal{G}}^A_B)^C_D(z,\bar{z})$ has conformal weights[5] $(h,\bar{h})$. Then we can expand the current as

$$
(\widetilde{\mathcal{G}}^A_B)^C_D(z,\bar{z}) = \sum_{n,m\in\mathbb{Z}} \{(\widetilde{\mathcal{G}}^A_B)^C_D\}_{mn} z^{-m-h}\bar{z}^{-n-\bar{h}}, \tag{5.88}
$$

with

$$
\{(\widetilde{\mathcal{G}}^A_B)^C_D\}_{mn} = \frac{1}{(2\pi i)^2}\oint dz \oint d\bar{z}\, z^{m+h-1}\bar{z}^{n+\bar{h}+1}(\widetilde{\mathcal{G}}^A_B)^C_D(z,\bar{z}). \tag{5.89}
$$

Suppose we had an OPE of the form[6]

$$
(\widetilde{\mathcal{G}}^A_B)^C_D(z,\bar{z})S_C(w) \sim \frac{1}{z-w}\frac{1}{\bar{z}-\bar{w}}(T^A_B)^C_D S_C(w). \tag{5.90}
$$

One can readily check that this would give us the commutator

$$
\left[(\widetilde{\mathcal{G}}^A_B)^C_D\}_{mn},(S_C)_k\right] = \bar{w}^n \left(T^A_B\right)^C_D \{S_C\}_{m+k} \tag{5.91}
$$

[5]the scaling dimensions of $(\widetilde{\mathcal{G}}^A_B)^C_D(z,\bar{z})$ can be calculated from those of $G_{AB},\overline{G}^{CD}$. It is $(a+b'-2,a'+b-2)$.
[6]exactly the same argument works if we have higher power singularities.

This is nonsensical since we do not have any $\bar{w}$ dependence on the left-hand side. Similarly one can justify the second part of (5.58). So we conclude that the $\mathcal{N}=8$ $\mathfrak{sbms}_4$ algebra does not contain the extension of global R-symmetry algebra. The final algebra is then given by

$$
\begin{aligned}
&[L_m, L_n] = (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} \\
&[L_n, P_{kl}] = \left(\frac{1}{2}n - k\right) P_{n+k,l}, \quad [\bar{L}_n, P_{kl}] = \left(\frac{1}{2}n - l\right) P_{k,n+l}, \\
&[L_n, (S_A)_m] = \left(\frac{n}{2} - m\right)(S_A)_{m+n}, \quad [\bar{L}_n, (S_A)_m] = 0 \\
&\left[L_n, \overline{S}^A_m\right] = 0, \quad \left[\bar{L}_n, \overline{S}^A_m\right] = \left(\frac{n}{2} - m\right)\overline{S}^A_{m+n} \\
&\{(S_B)_m, \overline{S}^A_n\} = \delta^A_B P_{mn}.
\end{aligned}
\tag{5.92}
$$

5.5 Conclusion

In this chapter, we have used the CCFT technique to compute the asymptotic symmetry algebra of $\mathcal{N}=8$ supergravity in asymptotically flat spacetime. The crucial part of our result is the non-extension of the global $SU(8)_R$ R-symmetry algebra. The purely mathematical considerations [390] for $\mathcal{N}=2$ theory suggests that the infinite-dimensional extension of R-symmetry is fraught with mathematical inconsistencies. Here, performing a direct asymptotic symmetry analysis of the supergravity theory using CCFT prescription, we have confirmed that, indeed, supergravity does not result in such an extension. The rest of the symmetry algebra is as expected and is presented in (5.92).

Let us end the chapter with relevant open problems. In the seminal work of Hawking et al., [141], the importance of the infinite number of soft hairs in the context of black hole microscopics was discussed. The study was further taken forward in [141, 333, 395–397] and beautifully reviewed in [11]. They emphasized the importance of symmetry enhancements at the future horizon $\mathcal{H}^+$ of the black holes and how both the hypersurfaces[7] $\mathcal{H}^+$ and $\mathcal{I}^+$ carry information of conserved charges, that are in turn important for understanding black hole microscopics. The study of the present work indicates that the asymptotic soft hairs of the supergravity theories will not have distinct infinite R-charges; rather, they will only carry the global fixed number of R-charges. An interesting question that remains to be studied is the effect, if any, of these R-charges at the horizon and, finally, their importance in the black hole microscopics. We hope to return to this question in the future.

[7] $\mathcal{I}^+$ denotes the future null horizon

In the next chapter, I will introduce a new prescription to study flat spacetime. Here, we will talk about the flat limit of AdS spacetime, where we are motivated to study the scattering processes of massive particles (scalars and spinning tensors).

CHAPTER 6

MASSIVE VECTOR PARTICLES SCATTERING IN ADS SPACETIME

This chapter is based on work (in progress) in collaboration with Nabamita Banerjee, Karan Fernandes, Arpita Mitra, and Amogh Desai.

6.1 Introduction

Our primary focus is to explore the consequences arising from the asymptotic behavior of particles within the AdS spacetime. Specifically, we aim to investigate how the *double scaling limit*[1] [239, 240] manifests itself in the correlation functions of the boundary CFT. As we have the relation between the flat space $\mathcal{S}$-matrix and AdS $\mathcal{S}$-matrix under the flat space limit ($R \to \infty$) of CFT correlators, we may now gain a better understanding of the soft theorems in AdS. This approach holds significant potential to unveil deeper insights into the AdS $\mathcal{S}$-matrix coming from the non-perturbative nature of boundary correlators.

In this study, we first take the approach of [236] to compute the “$\mathcal{S}$-matrix" of a QFT with charged particles in the flat space limit of AdS spacetime, and then we study the implications of the double scaling limit of [239] on them. Following [236], we get the $\mathcal{S}$-matrix via the Fourier transform of the position space correlation function in the embedding space, which is defined only for the on-shell momenta. This Fourier-transformed correlation function is interpreted as the AdS $\mathcal{S}$-matrix. Let us brief the steps for computing the "AdS $\mathcal{S}$-matrix" below :

- The momentum space correlation function is reduced to the flat space $\mathcal{S}$-matrix in the flat space limit. This implies that the correlation function will be proportional to the momentum-conserving delta function, $\delta(\sum P_i)$, at the leading order of $1/R$. As the translation symmetry is broken at orders of $1/R$ in the flat space limit for AdS potential, at the subleading order, we expect the correlation function to be proportional to the derivatives of $\delta(\sum P_i)$. Here, we define the AdS $\mathcal{S}$-Matrix as the term which is proportional to $\delta(\sum P_i)$ only, rather than its derivatives as proposed in [236].

- To compute the full n-point correlation, we make use of the Conformal Ward Identities or the action of the symmetry generators on our defined AdS $\mathcal{S}$-matrix at $\mathcal{O}(1/R^2)$. This

[1] *Double scaling limit* (DSL) introduced by Banerjee et al. [239, 240] is defined as a limit when the frequency (ω) of the radiation and the cosmological constant goes to zero (conversely, the AdS radius approaches infinity), simultaneously keeping their ratio constant. This limit physically uses the fact that when the space-time approaches to flat, these radiations become *soft*.

implies that the AdS $\mathcal{S}$-matrix has complete information on all of the bulk physics in terms of the conformal correlators in $1/R^2$ perturbations theory. We will further continue this discussion in section 6.4.

- In this work, we look at the exchange diagrams. In [236], authors showed the equivalence of this formalism with the Mellin space representation, which can be generalized to any generic correlation function using bulk-to-bulk and bulk-to-boundary propagators. Considering a bulk point in the patch, we compute the bulk to boundary propagator in $1/R$ perturbation theory, and this propagator is interpreted as the momentum space external leg factor for the Witten diagram. Moreover, we compute the Bulk to Bulk propagator, which can be used to compute higher-order diagrams.

To get the relevant expressions, we define a local patch on the AdS hyperboloid, which is centered around a specific local point $C^A = (\vec{0}, R)$, $A = 0, \cdots, d+1$, in the large R limit. This fixes a conformal frame for the momentum space correlator [236]. Once we fix the gauge for the AdS momenta variables, all the bulk processes are confined and localized within this frame surrounding the point C, where we represent the momenta variable as the flat space momenta and the isometry algebra of the flat space is defined around this point. This choice of gauge will become evident in the following sections. While taking the flat space limit $R \to \infty$, this point holds a distinct significance. Our work involves an extension of the previous work done by the authors in [236] in the context of vector fields.

The chapter is organized as follows. In section 6.2, we introduce the momentum space formalism for AdS $\mathcal{S}$-matrix as discussed in [236]. This will give a general prescription for scattering in AdS and how we will get the flat space $\mathcal{S}$-matrix under the flat space limit. Section 6.3.1 explains the construction of Bulk to Boundary propagator for massive vector fields under $R \to \infty$ limit. In section 6.3.2, we constructed the Bulk-to-Bulk propagator for the massive vector field up to sub-leading order and found the iterative solutions to the propagator equation of motion up to sub-leading order. In section 6.4, we explain the procedure to compute AdS $\mathcal{S}$-matrix and enlighten the reader about the other prescriptions that one can use. In section 6.5, we conclude the chapter with some motivation towards the application in the context of some low-energy effective field theories, which are instrumental considering the demand in theoretical studies.

6.2 Notation and Setup

This section aims to establish standardized notations and terminologies for subsequent discussions. Here, we closely follow the momentum space prescription for defining an AdS "$\mathcal{S}$-matrix" laid forward by [236]. The main idea is to work in the Fourier dual space of the embedding space coordinates for theories in AdS and the dual CFT. This has a few advantages, the most prominent being that the AdS isometries are manifest in this description.

Let us begin with the description of the embedding space of AdS spacetime. Both the AdS_{d+1} and its boundary dual CFT_d are embedded in a $\mathbb{R}^{2,d}$ space with metric $\widetilde{\eta}_{AB} \equiv diag(-,+,\ldots,+,-)$, $A = 0,\cdots,d+1$. Considering two vectors in the embedding space as X^A, W^A and R as the AdS radius, the hypersurface $\text{X}^2 = -R^2$ describes the AdS manifold and the dual boundary CFT lies on the null cone defined by $\text{W}^2 = 0$[2]. In this space, the AdS isometry generators are linear and are given by,

$$M_{AB} = \text{X}_A \frac{\partial}{\partial \text{X}^B} - \text{X}_B \frac{\partial}{\partial \text{X}^A}. \tag{6.1}$$

Similarly, the conformal symmetry generators are given by,

$$\widetilde{M}_{AB} = \text{W}_A \frac{\partial}{\partial \text{W}^B} - \text{W}_B \frac{\partial}{\partial \text{W}^A}. \tag{6.2}$$

Note that the structure of the two sets of generators are precisely identical, owing to the fact that the two manifolds possess the same isometry group, i.e., the rotation group $\text{SO}(d,2)$.

We now look at the Fourier transform of position-dependent correlation functions defined on the AdS manifold. We require that the Fourier dual function is defined only over the AdS manifold and has no contribution from the region of the embedding space beyond AdS. For this, we will demand that the Fourier integral over the entire embedding space has support only over the AdS manifold such that contribution to the Fourier integral due to regions away from the AdS vanishes.

We put such a restriction mathematically by introducing a $\delta(\text{X}^2 + R^2)$ in the integral. For an n-point position space correlation function $F(\text{X}_1,\cdots \text{X}_i,\cdots,\text{X}_n)$, we define its Fourier trans-

[2]CFTs are better understood by transforming to a space where the conformal group is realized as the group of linear isometries [398, 399]. This is accomplished by formulating CFTs on the projective null cone in two higher dimensions where it is only the action of the Lorentz group [400–402]. Consider a $d+2$ dimensional embedding space $\mathbb{R}^{d,2}$ with flat metric $ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = -dX^+ dX^- + dx^2$, where $X^\mu = (X^+, X^-, x^a)$, $a = 0,\cdots,d-1$. The null cone defined by $X^2 = 0$, is a $\text{SO}(d,2)$-invariant subspace of $\mathbb{R}^{d,2}$, and the projective null cone $\mathbb{P}\text{N}$ is obtained by quotienting by the scale $\Gamma = X.\partial_X$. Hence, $\mathbb{P}\text{N} = \left\{X \in \mathbb{R}^{d,2} | X^2 = 0\right\}/\Gamma$.

form $\widetilde{F}(\mathcal{P}_1,\cdots\mathcal{P}_i,\cdots,\mathcal{P}_n)$, where X_i's represents the position space vector insertions in the embedding space and $\mathcal{P}_i$'s are the Fourier dual variables for X_i. We have

$$\widetilde{F}(\mathcal{P}_1,\cdots\mathcal{P}_i,\cdots,\mathcal{P}_n) = \int\prod_{i=1}^{n} d^{d+2}\mathrm{X}_i\; \delta(\mathrm{X}_i^2+R^2)\; e^{i\mathcal{P}_i\cdot \mathrm{X}_i} F(\mathrm{X}_1,\cdots\mathrm{X}_i,\cdots,\mathrm{X}_n) \tag{6.3}$$

In the given expression, the integration is carried out over the AdS hyperboloid. Here, we use the shorthand notations for the correlations in position and momentum space as $F(\mathrm{X}_i)$ and $\widetilde{F}(\mathcal{P}_i)$ respectively, which are functions of $d+2$ variables. Note that the action of the symmetry generators is linear in nature, which means,

$$M_{AB}\widetilde{F}(\mathcal{P}_i) = \int\prod_{i=1}^{n} d^{d+2}\mathrm{X}_i\; \delta(\mathrm{X}_i^2+R^2) F(\mathrm{X}_i)\; M_{AB}\; e^{i\mathcal{P}_i\cdot \mathrm{X}_i}. \tag{6.4}$$

Here, the generator has bypassed the measure, the delta function, and the correlation function in position space because the measure must be invariant under symmetry transformation. Hence

$$[M_{AB},\mathrm{X}_i^2] = [M_{AB},F(\mathrm{X}_i)] = 0, \tag{6.5}$$

where the first commutator defines AdS isometry, and the second manifests the demand that the correlation functions must be invariant under symmetry transformations.

Substituting Eq. (6.1) in Eq. (6.4) we get,

$$\begin{aligned}
M_{AB}\widetilde{F}(\mathcal{P}_i) &= \int\prod_{i=1}^{n} d\mathbf{X}_i\; \delta({\mathrm{X}_\mathrm{i}}^2+R^2)F(\mathrm{X}_i)\Big(\mathrm{X}_A\mathcal{P}_B-\mathrm{X}_B\mathcal{P}_A\Big)e^{i\mathcal{P}_i\cdot \mathrm{X}_i}\\
&= \int\prod_{i=1}^{n} d\mathbf{X}_i\; \delta(\mathrm{X}_i^2+R^2)F(\mathrm{X}_i)\Big(\mathcal{P}_B\frac{\partial}{\partial\mathcal{P}_A}-\mathcal{P}_A\frac{\partial}{\partial\mathcal{P}_B}\Big)e^{i\mathcal{P}_i\cdot \mathrm{X}_i}\\
&= \Big(\mathcal{P}_B\frac{\partial}{\partial\mathcal{P}_A}-\mathcal{P}_A\frac{\partial}{\partial\mathcal{P}_B}\Big)\int\prod_{i=1}^{n} d\mathrm{X}_i\; \delta(\mathrm{X}_i^2+R^2)\; F(\mathrm{X}_i)e^{i\mathcal{P}_i\cdot \mathrm{X}_i}\\
&= \Big(\mathcal{P}_B\frac{\partial}{\partial\mathcal{P}_A}-\mathcal{P}_A\frac{\partial}{\partial\mathcal{P}_B}\Big)\;\widetilde{F}(\mathcal{P}_i)
\end{aligned} \tag{6.6}$$

Hence, the symmetry generators in momentum space up to an overall negative sign are given by,

$$M_{AB} = \Big(\mathcal{P}_A\frac{\partial}{\partial\mathcal{P}_B}-\mathcal{P}_B\frac{\partial}{\partial\mathcal{P}_A}\Big). \tag{6.7}$$

Substituting the Fourier transformed correlator of Eq 2.3 in the LHS of the above relation, one can also check that the momentum space correlation function satisfies the property,

$$\left(\frac{\partial}{\partial \mathcal{P}_i}\cdot\frac{\partial}{\partial \mathcal{P}_i}-R^2\right)\widetilde{F}(\mathcal{P}_i)=0 \tag{6.8}$$

This is a constraint on the momentum space correlators, which is used to reduce the degrees of freedom from $d+2$ to $d+1$. This can be incorporated by fixing $\mathcal{P}_{i,d+1}=0$.[3] Then $\widetilde{F}$ is an unconstrained function of the $d+1$ variables $\mathcal{P}_{i,a}$, $a=0,1,\ldots,d$. This momentum is now unconstrained and will henceforth be denoted by P_a. We will refer to this as the “AdS momentum". The momentum space symmetry generators can now be written as,

$$M_{ab}=P_a\frac{\partial}{\partial P^b}-P_b\frac{\partial}{\partial P^a},\quad M_{a,d+1}=P_a\frac{\partial}{\partial \mathcal{P}^{d+1}}-\mathcal{P}_{d+1}\frac{\partial}{\partial P^a}=iP_a\sqrt{R^2-\frac{\partial}{\partial P}\cdot\frac{\partial}{\partial P}}$$

Notice that, in the flat space limit ($R\to\infty$), $M_{a,d+1}$ become translation symmetry generators (or momentum P_a) up to a factor of R. Moreover M_{ab} behave as generators of rotation in the X_a-X_b plane and since $a,b=0,1,\ldots,d$, we have $(d+1)d/2$ such generators. We refer the readers to Appendix D.2 for the algebra of the AdS symmetry group in the large R limit. Thus, we recover the Poincaré group from the AdS isometry group in the flat space limit of AdS. This was expected because in the flat space limit of AdS, we must be able to recover all the features and physics of a flat space from equivalent features of AdS.

In $3+1$ dimensions, the Poincaré group has two Casimirs, namely, mass and spin, which have two distinct eigenvalues. However, in the presence of AdS potential, there exists only one physically relevant Casimir whose eigenvalues can be written as $M^2R^2=\Delta(\Delta-d)+l(l+d-2)$, where $\Delta(\Delta-d)$ corresponds to the mass and $l(l+d-2)$ as the spin part of the Casimir in Poincaré group. Hence, AdS Casimir has information about both mass and spin, which was expected as translation symmetry breaks down due to the AdS potential.

6.2.1 Momentum Space CFT Correlation in flat space limit

Here our objective is to retrieve the flat spacetime momentum signature. To accomplish this, it is essential to constrain the results on AdS hyperboloid at the slice $\mathcal{P}_{i,d+1}=0$. By doing so, we can precisely capture and analyze the momentum vector signature in the context of the flat space momentum.

In the large R limit, we have the coordinates defined in the local patch around C as X_a,

[3]There can be other such choices, but this seems to be the simplest one and works well for our purpose [236].

$a = 0, \cdots, d$, and the local momentum defined to be P_a. The embedding space coordinates are $\mathrm{X} = (X_a, (R^2 + X_a X^a)^{1/2})$. After integrating out the X_{d+1} integral in Eq. (6.3), the bulk n-point correlation function can be written as,

$$\widetilde{F}(P_a) = \int \prod_{i=1}^{n} \frac{d^{d+1} X_{a,i}}{\sqrt{R^2 + X_{a,i} X_i^a}} e^{iP_{a,i} \cdot \mathrm{X}_{a,i}} F(X_a, (R^2 + X_{a,i} X_i^a)^{1/2}) \tag{6.9}$$

This transformation is defined for the off-shell momenta P_a. Here, we need to define a Fourier transform for the on-shell momenta such that the CFT operator insertions behave as the *on-shell* asymptotic states for scattering in AdS. On account of this, we call this momentum space CFT correlation to be the “AdS $\mathcal{S}$-matrix".

In AdS/CFT, using the “extrapolate" dictionary [403, 404], we can relate the bulk field $\phi(\mathrm{X})$ in AdS to the boundary CFT operator $\mathcal{O}(\mathrm{W})$ with conformal dimension Δ as,

$$\mathcal{O}_\Delta(\mathrm{W}) = \lim_{\alpha \to \infty} \alpha^\Delta \phi(\mathrm{X} = \alpha W + O(\alpha^0)). \tag{6.10}$$

Here W is the embedding space vector labeling a point at the boundary such that $\mathrm{W}^2 = 0$, and X is a point in AdS satisfying $X^2 + R^2 = 0$. $O(\alpha^0)$ can be a vector in AdS hyperboloid orthogonal to the lightcone vector W [236].

Now, the Fourier transform of the boundary CFT correlation function $G(\mathrm{W}_i)$ following the above dictionary can be written as,

$$\widetilde{G}(\mathcal{P}_i) = \int \prod_{i=1}^{n} d^{d+2} \mathrm{W}_i \, \delta(\mathrm{W}_i^2) \, e^{i\mathcal{P}_i \cdot \mathrm{W}_i} \, G(\mathrm{W}_i) \tag{6.11}$$

This Fourier integral looks the same as in Eq.(6.3) except the support, which is only over the CFT manifold, which is the projective null cone, $\mathrm{W}_i^2 = 0$ other than the AdS hyperboloid. This function $\widetilde{G}(\mathcal{P}_i)$ satisfies,

$$\frac{\partial}{\partial \mathcal{P}_i} \cdot \frac{\partial}{\partial \mathcal{P}_i} \widetilde{G}(\mathcal{P}_i) = 0. \tag{6.12}$$

As we have seen, this condition imposes a constraint on $\mathcal{P}$, which allows us to set $\mathcal{P}_{d+1} = 0$ and $\mathcal{P} = (P_a, 0)$ like in AdS.

Let’s define a scale covariant function $\widehat{G}(\hat{\mathcal{P}}_i)$ in momentum space, which is a function of the unit momentum vector $\hat{\mathcal{P}}_i \equiv \mathcal{P}_i / |\mathcal{P}_i|$. The conformal covariance of the CFT correlation function in Fourier space implies

$$\widehat{G}(\hat{\mathcal{P}}_i) = \widetilde{G}(\mathcal{P}_i)\left(|\mathcal{P}_i|^{d-\Delta_i}\right) \tag{6.13}$$

It's clear that as long as $\Delta \neq d$, i.e., for massive particles in the bulk of AdS, this new function depends only on the unit vector $\hat{\mathcal{P}}_i$.

We have to fix the gauge[4] $|\mathcal{P}|^2 = -M^2 = |P_a|^2$, such that the correlation function will be a function of only on-shell AdS momenta P_a. Hence, we can construct the on-shell asymptotic states with the support of the Fourier transform only over the boundary.

This analysis implies that the on-shell asymptotic states are constructed by the Fourier transform of the boundary data of the AdS space [236]. So, whenever we compute correlation functions with the conformal momenta constraint $\mathcal{P}^2 = -M^2$, they will be "on-shell", and in the flat space limit of these AdS/CFT correlation functions, we should be able to retrieve flat space $\mathcal{S}$-matrix elements. It has been shown explicitly for some scalar Witten diagrams [236]. We have extended this to massive vector fields. In momentum space, the boundary n-point function is characterized as the Fourier transform of the corresponding position space n function. In the limit $R \to \infty$, this correlation gives us the conformal momentum space observables in the flat space.

To see the effect of the large R, we have to scale the vector $\mathrm{W}_i \to R\mathrm{W}_i$ in Eq (6.11), and fix the gauge by using the condition, $\mathcal{P}_{d+1} = 0$. Here, the integration support for both W_1 and W_2 are on the future null cone[5]. The explicit derivation of this correlation is given in Appendix [236, A.1].

6.3 Proca theory in AdS$_{d+1}$

In this section, we will consider a massive vector field in AdS and derive its bulk-to-boundary and bulk-to-bulk propagators in momentum space. The AdS$_{d+1}$ spacetime can be described by embedding it in $(d+2)$ dimensional Minkowski spacetime $\eta_{AB} = \mathrm{diag}(-,+,\cdots,-)$. The coordinates of AdS_{d+1} of length R is a set of points $X \equiv (X^0, X^1, \cdots, X^d)$ and the spacetime

[4]Since this function does not depend on the norm of $\mathcal{P}_i$, we can gauge fix the momenta using the condition, $|\mathcal{P}_i|^2 = -M^2 = |P_i|^2$. Proper justification for this choice of gauge is given using conformal Casimir and from the boundary to bulk propagator in [236, Section 3.1]

[5]$\delta(W_i^2)$ in the integrand tells us that we restrict the non-zero contributions of the Fourier integral over the null cone.

interval is given by,

$$ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \cdots + \left(dX^d\right)^2 - \left(dX^{d+1}\right)^2 \tag{6.14}$$

In embedding coordinates, AdS_{d+1} is a hyperboloid satisfying $\mathcal{X}^A\mathcal{X}_A = -R^2$ with its boundary given by the projective null cone $\mathcal{W}^A\mathcal{W}_A = 0$. To distinguish the AdS$_{d+1}$ spacetime metric, we define $W = X^{d+1}$ such that Eq.(6.14) takes the form

$$ds^2 = \eta_{\mu\nu}dX^\mu dX^\nu - dW^2 \tag{6.15}$$

The hyperboloid constraint condition further implies that $W^2 = R^2 + X^2$, with $X^2 = X_\mu X^\mu$. Since we are interested treating $1/R$ as a perturbative parameter and studying corrections up to $1/R^2$ order, we will consider the metric and inverse metric as,

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{X_\mu X_\nu}{R^2 + X^2}, \quad g^{\mu\nu}(X) = \eta^{\mu\nu} + \frac{X^\mu X^\nu}{R^2} + \mathcal{O}\left(R^{-3}\right) \tag{6.16}$$

The covariant derivatives and the affine connection are given by

$$\nabla_X = \partial_X - \Gamma, \quad \Gamma^\gamma_{\alpha\beta} = -\frac{1}{\mathrm{R}^2}X^\gamma g_{\alpha\beta}. \tag{6.17}$$

We note that the connection in Eq.(6.17) is exact to all orders in R^{-1}. We will now consider a massive vector field A^μ with mass M in AdS$_{d+1}$, which is described by the Proca action on the AdS background [405, 406],

$$S_M = -\int d^{d+1}X\sqrt{-g}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu\right], \tag{6.18}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The action provides the following equation of motion for A^μ,

$$\nabla_\mu F^{\mu\nu} - M^2 A^\nu = 0, \tag{6.19}$$

which on using Eq.(6.16) and (6.17) gives [6],

$$\left[g_{\mu\nu}(\nabla_X^2 - M^2) - \nabla_\nu\nabla_\mu\right]A^\nu = \partial_\nu A^\nu - \frac{1}{R^2}X^\nu\eta_{\nu\rho}A^\rho + \mathcal{O}(R^{-3}) = 0. \tag{6.20}$$

This equation of motion will be used to derive the bulk-to-bulk and bulk-to-boundary propagators for the massive vector field. Lastly, we note that the Proca field by virtue of Eq.(6.20) further satisfies the condition

$$\nabla_\mu A^\mu = 0. \tag{6.21}$$

6.3.1 Bulk-to-boundary propagator

In general for AdS spacetimes in the embedding formalism, the bulk-to-boundary propagator between two fields inserted at a local bulk point $\mathcal{X}$ and a point on boundary $\mathcal{W}$ is proportional to $(\mathcal{W}\cdot\mathcal{X})^{-\Delta}$. For vector fields, we have [398, 406–408].

$$E_\Delta^{\nu\beta}(\mathcal{X},\mathcal{W}) = \left(\eta^{\nu\beta} - \frac{\mathcal{W}^\nu\mathcal{X}^\beta}{(\mathcal{W}\cdot\mathcal{X})}\right)\frac{1}{(-2\mathcal{W}\cdot\mathcal{X})^\Delta} \tag{6.22}$$

The corresponding expression in momentum space can be derived by performing an appropriate Fourier transform on the boundary coordinate, i.e.

$$\widetilde{\Pi}^{\nu\beta}(\mathcal{P},\mathcal{X}) = \int d\mathcal{W}\delta(\mathcal{W}^2)e^{i\mathcal{P}\cdot\mathcal{W}}E_\Delta^{\nu\beta}(\mathcal{X},\mathcal{W}) \tag{6.23}$$

We now recall some salient properties of (6.23) that will provide for external states in the AdS S-matrix with fixed momenta. These are identical to properties noted in [236] for the bulk-to-boundary propagator for massive scalar fields. From the homogeneity property $(\lambda\mathcal{W}\cdot\mathcal{X})^{-\Delta} = \lambda^{-\Delta}(\mathcal{W}\cdot\mathcal{X})^{-\Delta}$, it follows that

$$\begin{aligned} E_\Delta^{\nu\beta}(\mathcal{X},\lambda\mathcal{W}) = \lambda^{-\Delta}E_\Delta^{\nu\beta}(\mathcal{X},\mathcal{W}) &\Rightarrow \tilde{\Pi}^{\nu\beta}(\lambda\mathcal{P},\mathcal{X}) = \lambda^{\Delta-d}\tilde{\Pi}^{\nu\beta}(\mathcal{P},\mathcal{X}) \\ &\Rightarrow \tilde{\Pi}^{\nu\beta}(\hat{\mathcal{P}},\mathcal{X}) = |\mathcal{P}|^{d-\Delta}\tilde{\Pi}^{\nu\beta}(\mathcal{P},\mathcal{X}) \end{aligned} \tag{6.24}$$

The last transformation in (6.24) ensures that we can derive vector bulk-to-boundary prop-

[6] This also follows from the group theoretic point of view when one considers that the AdS Casimir $M^2R^2 = \Delta(\Delta-d)+l(l+d-2)$ acts as the Laplacian in embedding space coordinates whose eigenvalue is the mass of the field.

agators as a function of the orientation $\hat{\mathcal{P}} = \frac{\mathcal{P}}{|\mathcal{P}|}$ while fixing the magnitude $|\mathcal{P}|$. In the case of massive fields setting $\mathcal{P}_{d+1} = 0$ such that $\mathcal{P}^2 = -M^2 = P^2$. We note that this property follows from the leading contribution of (6.23) coming from $\mathcal{W} = (W,1)$, with $W^{\mu} = i\frac{P^{\mu}}{M}$ in the large R limit as noted in [223]. In this way, the formal Fourier transform to "momentum" space in embedding coordinates can be restricted to fixed on-shell momenta states.

Let us here discuss the massless limit briefly. Generically the massless limit is defined as $\Delta \to J+d-2$ for any spin-J particle [406]. Thus for spin 1 massless limit we have $\Delta \to 1, d-1$. In this case, the above scaling relation still holds with $\hat{\mathcal{P}}$ being the direction of the mull vector that also has magnitude zero.

In setting $\mathcal{P}_{d+1} = 0$, the embedding position takes the form $\mathcal{X} = \left(X\,,\sqrt{X^2+R^2}\right)$ and this provides a perturbative expansion in R^{-1} about a flat spacetime patch in the strict $R \to \infty$ limit. Henceforth, we will consider this case and denote the relevant bulk-to-boundary propagator of (6.23) by $\widetilde{\Pi}^{\nu\beta}(P,X)$. We can recover asymptotic plane wave states from (6.23) in the flat patch to leading order in R^{-1} by working in the limit of large $\Delta = MR \gg 0$. On rescaling $\mathcal{W} \to R\mathcal{W}$ and using $\mathcal{W} = (W,1)$ in (6.23) we get

$$\begin{aligned}\widetilde{\Pi}^{\nu\beta}\left(P\,,X\right) =& \frac{R^{\Delta-d}}{2\pi}\int d\mathcal{W}\int d\lambda\, e^{iR\lambda\left(W^2-1\right)}e^{iRP.W} \\ &\left(\eta^{\nu\beta} - \frac{\mathcal{W}^{\nu}\mathcal{X}^{\beta}}{(\mathcal{W}\cdot\mathcal{X})}\right)(-2)^{-MR}\left(W\cdot X - \sqrt{X^2+R^2}\right)^{-MR}\,, \end{aligned} \tag{6.25}$$

with a λ integral representation for the Dirac delta function. In the large R limit, the integrand contribution from the second line of (6.25) is approximated by

$$\left(W\cdot X - \sqrt{X^2+R^2}\right)^{-MR} \approx (-R)^{-MR}e^{MW.X} \tag{6.26}$$

which notably is not exponential in R, unlike the integrand contribution in the first line of (6.25). The exponential terms provide the saddle contribution $W_*^{\mu} = i\frac{P^{\mu}}{m}$ in the large R limit. Using this saddle contribution in 6.25 and using 6.26, we get

$$\widetilde{\Pi}^{\nu\beta}\left(P\,,X\right) \propto \alpha^{\nu\beta}e^{iP.X}\,. \tag{6.27}$$

Hence in the large R limit, the leading contribution from the Fourier transformed bulk-to-boundary propagator are plane wave states. However, the derivation of perturbative corrections in R^{-1} from the Fourier transform, which will come from expanding $\mathcal{X}$, is more involved using the saddle approximation about large R to the Fourier transform.

A simpler alternative towards perturbative R^{-1} corrections in the bulk-to-boundary propagator comes directly from the source-free equations of motion that it satisfies in embedding space. We will solve the propagator in the large R limit with an iterative treatment to derive R^{-1} perturbations about the leading flat space limit. From Eq.(6.20), the bulk-to-boundary propagator $\widetilde{\Pi}^{\nu\beta}(P;X)$ satisfies the following differential equation[7],

$$\left[g_{\mu\nu}\left(\nabla_X^2 - M^2\right) - \nabla_\nu\nabla_\mu\right]\widetilde{\Pi}^{\nu\beta}(P;X) = 0. \tag{6.28}$$

After a bit of simplification using the relation in D.11, we have

$$\left[\left(\eta_{\mu\nu} - \frac{X_\mu X_\nu}{X^2+R^2}\right)(\nabla_X^2 - M^2) - \partial_\mu\partial_\nu + \frac{X_\nu\partial_\mu}{R^2} + \frac{(d+2)\eta_{\mu\nu}}{R^2}\right]\Pi^{\nu\beta} + \frac{X^\beta\eta_{\rho\sigma}\partial_\mu\Pi^{\rho\sigma}}{R^2} = 0, \tag{6.29}$$

with the embedding space, Laplacian has the following expansion

$$\nabla_X^2\Pi^{\nu\beta} = g^{\rho\sigma}\nabla_\rho\nabla_\sigma\Pi^{\nu\beta} = \partial^2\Pi^{\nu\beta} - \frac{1}{R^2}\Big[-X^\rho X^\sigma\partial_\rho\partial_\sigma\Pi^{\nu\beta} + 2\Pi^{\nu\beta} + 2X^\nu\partial_\alpha\Pi^{\alpha\beta} + 2X^\beta\partial_\alpha\Pi^{\nu\alpha} - (d+1)X^\alpha\partial_\alpha\Pi^{\nu\beta}\Big].$$

As noted previously, the bulk-to-boundary propagator can be derived as a function of the momentum direction. We accordingly define $\eta = i\hat{P}\cdot X$, with the unit momentum vector $\hat{P}_i = \frac{P_i}{|P_i|}$ and derive the solution for $\widetilde{\Pi}^{\nu\beta}(\eta)$. The perturbative solution of Eq.(6.29) then takes the form

$$\widetilde{\Pi}^{\nu\beta}(\eta) = e^{iM\eta}\left[f_1^{\nu\beta} + \frac{1}{R^2}f_2^{\nu\beta} + \mathcal{O}\left(R^{-3}\right)\right]. \tag{6.30}$$

It is evident from Eq.(6.29) that the propagator has two solutions depending on the choice $P = \pm iM\hat{P}$. On physical grounds, we discard the negative energy solution from the onset. Therefore we set $e^{iM\eta} = e^{iP\cdot X}$. We have provided the detailed derivations of the leading and subleading contributions of 6.30 in Appendix D.3. In the following, we summarize the main steps and quote the results.

[7] Traditionally, there is a delta function on the R.H.S. of the above Green's function equation given in Eq.(6.28). However, the boundary field operator is in the momentum space, so one should have expected it to contain a $e^{iP\cdot X}$ term. Notice that the two field insertions lie on different manifolds, meaning their positions never coincide. As a result, the delta function becomes redundant, and there is no phase factor.

Substituting Eq.(6.30) in Eq.(6.29), we obtain the leading order equation,

$$\left[\eta_{\mu\nu}(\partial_X^2 - M^2) - \partial_\mu\partial_\nu\right] e^{iM\eta} f_1^{\nu\beta} = 0. \tag{6.31}$$

Eq.(6.31) can be readily solved to get

$$f_1^{\nu\beta} = \eta^{\nu\beta} - \hat{P}^\nu\hat{P}^\beta, \qquad \eta_{\mu\nu} f_1^{\mu\nu} = d, \qquad P_\nu f_1^{\nu\beta} = 0. \tag{6.32}$$

To next order, we have an equation with a $1/R^2$ correction, which follows the perturbative expansion of Eq.(6.29)

$$\begin{aligned}\Big[\eta_{\mu\nu}(\partial^2 - M^2) - \partial_\mu\partial_\nu\Big] f_2^{\nu\beta} e^{iM\cdot\eta} + f_1^{\nu\beta}\Big[\eta_{\mu\nu}\Big(d + (X\cdot\partial_X)^2 + (d+1)X\cdot\partial_X\Big) + X_\nu\partial_\mu\Big] e^{iM\cdot\eta} \\ - \Big\{2\eta_{\mu\nu}\Big(X^\nu f_1^{\alpha\beta}\partial_\alpha + X^\beta f_1^{\alpha\nu}\partial_\alpha\Big) - dX^\beta\partial_\mu\Big\} e^{iM\cdot\eta} = 0\end{aligned} \tag{6.33}$$

We can solve this equation by substituting the leading order solution for $f_1^{\nu\beta}$, which acts as an effective source for the first order perturbed solution $f_2^{\nu\beta}$. Additionally, we assume a general ansatz for $f_2^{\nu\beta}$ as a linear combination of all possible two-index tensor quantities, i.e. $\eta^{\nu\beta}, P^\nu P^\beta, X^\nu X^\beta, P^\nu X^\beta, X^\nu P^\beta$. The solution for $f_2^{\nu\beta}$ in Eq.(6.33) then follows from solutions for the coefficients in the linear combination ansatz, and we find

$$\begin{aligned} f_2^{\nu\beta} = &\left[-\frac{(d+2)(\hat{P}\cdot X)}{4M} - \frac{d(\hat{P}\cdot X)^2}{4} + \frac{M(\hat{P}\cdot X)^3}{6}\right] f_1^{\nu\beta} \\ &- \frac{(d+1)}{M} X^\beta\hat{P}^\nu + \frac{1}{M}X^\nu\hat{P}^\beta + \frac{d}{M^2}\hat{P}^\beta\hat{P}^\nu. \end{aligned} \tag{6.34}$$

Hence, the bulk-to-boundary propagator as a function of (P,X) up to the first subleading correction in R^{-2} takes the form

$$\begin{aligned}\widetilde{\Pi}^{\nu\beta}(P,X) = e^{iP.X}\Bigg[& f_1^{\nu\beta} + \frac{1}{R^2}\left(\frac{i(d+2)}{4M^2}(P\cdot X) + \frac{d}{4M^2}(P\cdot X)^2 + \frac{i}{6M^2}(P\cdot X)^3\right) f_1^{\nu\beta} \\ & + \frac{1}{R^2}\left(\frac{i(d+1)}{M^2}X^\beta P^\nu - \frac{i}{M^2}X^\nu P^\beta - \frac{d}{M^4}P^\beta P^\nu\right)\Bigg]\end{aligned} \tag{6.35}$$

While a direct $M \to 0$ limit results in a divergence in (6.34), it is interesting to note a possible double scaling approach that could address the massless limit. If we take $R \to \infty$ and $M \to 0$, keeping the product $MR = \gamma$ as a large fixed number while also retaining the orientation $\hat{P}$ fixed,

$$\widetilde{\Pi}^{\nu\beta} = e^{i\hat{P}.X}\left[f_1^{\nu\beta} + \frac{d}{\gamma^2}\hat{P}^{\beta}\hat{P}^{\nu}\right]. \tag{6.36}$$

However, we would now require the scaling dimension of the boundary field to satisfy

$$\Delta(\Delta - d) = \gamma^2 - (d-1), \tag{6.37}$$

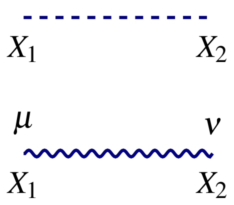
which implies Δ has two solutions. By imposing unitarity we get,

$$\Delta = \frac{d}{2} + \frac{1}{2}\sqrt{(d-2)^2 + 4\gamma^2}. \tag{6.38}$$

Hence, the case of setting $MR = \gamma$ and allowing for a simultaneous massless and flat spacetime limit differs from the conventional assumption for massless vector fields in AdS, namely that they are dual to CFT operators with conformal dimensions either 1 or $d-1$. Next, we consider the derivation of the bulk-to-bulk propagator for massive vector fields.

6.3.2 Bulk-to-Bulk propagator

The bulk-to-bulk propagator is a two-point correlation function of a given field theory, interpreted physically as the amplitude for its propagation between any two points X and Y in the bulk of AdS. We will be interested in the propagators for massive vector and scalar fields, which are defined using the time-ordered product (T-product).



$$: \quad G(X_1, X_2) \equiv \langle T\{\Phi(X_1)\Phi(X_2)\}\rangle.$$

$$: \quad \Pi^{\mu\nu}(X_1, X_2) \equiv \langle T\{A^{\mu}(X_1)A^{\nu}(X_2)\}\rangle.$$

Figure 6.1: Definition of vector and scalar propagators in Bulk AdS.

The momentum space bulk-to-bulk vector propagator, with one of the positions Fourier transformed to momentum space, will be important in our consideration of Witten diagrams. The bulk-to-bulk propagator $g_{BB}(\tilde{P}, X)$ for a scalar field in AdS involving R^{-2} corrections was derived in [236]. While $\tilde{P}$ is defined about a locally flat region within AdS in the large R limit, it need not be on-shell as in the bulk-to-boundary propagator case. To derive $G^{\mu\nu}_{BB}(\tilde{P},X)$ for the massive vector field, we work with the equations of motion satisfied by the position space bulk-to-bulk propagator

$$\Big[g_{\mu\nu}(\nabla^2_X - M^2) - \nabla_\nu\nabla_\mu\Big] G^{\nu\beta}_{BB}(Y,X) = -\frac{1}{\sqrt{g(X)}}\delta^\beta_\mu \delta(Y,X), \tag{6.39}$$

and Fourier transform Y to momentum space to find

$$\Big[g_{\mu\nu}(\nabla^2_X - M^2) - \nabla_\nu\nabla_\mu\Big] G^{\nu\beta}_{BB}(P,X) = -\frac{1}{\sqrt{g(X)}}\delta^\beta_\mu e^{iP\cdot X} \tag{6.40}$$

As in the derivation of the bulk-to-boundary propagator, we consider $G^{\nu\beta}_{BB}$ perturbatively expanded in powers of R^{-2} such that

$$G^{\nu\beta}_{BB} = e^{iP\cdot X}\left[G^{\nu\beta}_{BB(1)} + \frac{1}{R^2}G^{\nu\beta}_{BB(2)} + \dots\right] \tag{6.41}$$

The leading contribution can be derived using the ansatz

$$G^{\nu\beta}_{BB(1)}(P,X) = (A(P,X)\eta^{\nu\beta} + B(P,X)P^\nu P^\beta)\, e^{iP.X}.$$

On substituting this in the leading term of Eq. 6.40, we find the solution

$$G^{\nu\beta}_{BB(1)}(P,X) = \left(\eta^{\nu\beta} + \frac{P^\nu P^\beta}{M^2}\right)\frac{e^{iP\cdot X}}{P^2+M^2}, \tag{6.42}$$

which is the leading contribution of the bulk-to-bulk massive vector propagator in a locally flat region of the AdS spacetime. The equation to solve for its subleading R^{-2} correction follows from using Eq. 6.41 in Eq.(6.40), which provides

$$\begin{aligned}&\Big[\eta_{\mu\nu}(\partial^2 - M^2) - \partial_\mu\partial_\nu\Big] e^{iP\cdot X}G^{\nu\beta}_{BB(2)} - 2\eta_{\mu\nu}X^\nu(\partial_\alpha e^{iP\cdot X})G^{\alpha\beta}_{BB(1)}\\ &+\Big[-X_\mu X_\nu(\partial^2 - M^2) + \eta_{\mu\nu}\Big(d + (X\cdot\partial_X)^2 + dX\cdot\partial_X\Big) + X_\nu\partial_\mu\Big] e^{iP\cdot X}G^{\nu\beta}_{BB(1)}\\ &-X^\beta(\partial_\alpha e^{iP\cdot X})G^{\alpha\nu}_{BB(1)} + \eta_{\rho\sigma}X^\beta(\partial_\mu e^{iP\cdot X})G^{\rho\sigma}_{BB(1)} = -\delta^\beta_\mu\frac{X\cdot X}{2}e^{iP.X}.\end{aligned} \tag{6.43}$$

We can solve this equation by introducing an ansatz for $G^{\nu\beta}_{BB(2)}$ as a linear combination of all possible two index tensors ($X^\mu X^\nu$, $P^\mu P^\nu$, $X^\mu P^\nu$, $X^\nu P^\mu$ and $\eta^{\mu\nu}$). The coefficients are scalar functions of (X,P), which include X^2, $X.P$, P^2 and their powers. On account of the source terms, from X^2 and those from the leading solution $G^{\nu\beta}_{BB(1)}$, the most general ansatz for $G^{\nu\beta}_{BB(2)}$ needed to solve Eq. (6.43) takes the form

$$G^{\nu\beta}_{BB(2)}(P,X) = \frac{e^{iP\cdot X}}{P^2+M^2}\Bigg(C(P,X)\eta^{\nu\beta} + X^\nu X^\beta + D(P,X)\, X^\nu P^\beta + E(P,X)\, X^\beta P^\nu + F(P,X)\, P^\nu P^\beta + \frac{X\cdot X}{2}\left(\eta^{\nu\beta} + \frac{P^\nu P^\beta}{M^2}\right)\Bigg) \tag{6.44}$$

with $C(P,X)$, $D(P,X)$, $E(P,X)$ and $F(P,X)$ the coefficients. Using Eq.(6.43), we find the following coefficient solutions

$$C(P,X) = -\frac{(P\cdot X)^2}{(P^2+M^2)} + \frac{i(P\cdot X)\big((2+d)M^4+(-3+d)M^2P^2-P^4\big)}{M^2(P^2+M^2)^2} + \frac{\big(-8M^2P^2+d(M^4-P^4)\big)}{(P^2+M^2)^3} \tag{6.45}$$

$$F(P,X) = -\frac{(P\cdot X)^2}{M^2(P^2+M^2)} + \frac{i(P\cdot X)\big((7+d)M^2+(3+d)P^2\big)}{M^2(P^2+M^2)^2} + \frac{(8+3d)M^4+4dM^2P^2+dP^4}{M^2(P^2+M^2)^3} \tag{6.46}$$

$$D(P,X) = \frac{(-i+P\cdot X)}{M^2} = E(P,X) \tag{6.47}$$

Substituting Eqs.(6.45-6.47) in Eq.(6.44), we get the desired bulk-to-bulk vector propagator solution up to subleading R^{-2} corrections. In the following section, we will use the propagators in the computation of Witten diagrams and

6.4 AdS $\mathcal{S}$-matrix

In the previous two sections, we computed the two propagators needed to compute correlation functions in the boundary CFT. Following the prescription given in [236] for massive scalar fields, in this section, we will define an “AdS $\mathcal{S}$-matrix" for massive vector fields from the correlation functions.

In flat space QFTs, the protocol for computing the $\mathcal{S}$-matrix for vector fields consists of applying LSZ prescription on the momentum space correlation functions and contracting them with polarization vector to get the final $\mathcal{S}$-matrix element. In doing so, one keeps in mind the transversality of the external momenta and the polarization vector. Here, we follow the same prescription considering a theory of a massive vector field with a perturbation to a flat space with a small negative cosmological constant.

We have the bulk $\mathcal{S}$-matrix defined as the n-point correlation,

$$\mathcal{A}(P_1,P_2,\ldots,P_n)=\varepsilon_{\mu_1}(P_1)\varepsilon_{\mu_2}(P_2)\ldots\varepsilon_{\mu_n}(P_n)\,F^{\mu_1\mu_2\ldots\mu_n}(P_1,P_2\ldots P_n) \tag{6.48}$$

where $\varepsilon_i(P_i)$ is the polarization corresponding to external momenta P_i and $F^{\mu_i}(P_i)$ is the correlation function that is computed using the Witten diagrams and the propagators in Eq.(6.34) and Eq.(6.44). Notice that we haven't defined an LSZ prescription in this procedure as the external particles are already on-shell, $P_i^2=-M^2$. Now, using the shorthand notation $\mathcal{A}(P_i)$, the quantity in Eq.(6.48) has complete information about the *scattering* process. As in [236] and as mentioned before, this "$\mathcal{S}$-matrix" is invariant under the action of symmetry generators M_{AB}.

After contracting the correlation function with polarization vectors, the resultant quantity $\mathcal{A}(P_i)$ can be expanded as,

$$\mathcal{A}(P_i)=\int dX^{d+1}\Big[A_{(0)}(P_i,\varepsilon_i)+[A_{(1)}(P_i,\varepsilon_i)]^{\mu_1}X_{\mu_1}+[A_{(2)}(P_i,\varepsilon_i)]^{\mu_2\mu_3}X_{\mu_2}X_{\mu_3}+\ldots\Big]\prod_i^n e^{iP_i\cdot X} \tag{6.49}$$

where $A_{(k)}(P_i,\varepsilon_i)$ generates k derivative terms acting on the momentum-conserving delta function in the momentum space. The first term in the above expression is proportional to a delta function of momenta, while the other terms are proportional to the derivatives of the delta function.

Here, one can see the explicit dependence of X factors in the amplitude. These terms are expected as compared to the flat space QFT results. In the flat space case, the delta function of momentum is the manifestation of space-time translation symmetry. In the case of AdS, the same is broken, and hence, in our result, these factors show up at $\mathcal{O}(1/R^2)$.

The X^μ in this expression can be replaced with $-i\partial_{P_n}$ acting on the exponential.

$$\mathcal{A}(P_i)=\int dX^{d+1}\Big[A_{(0)}(P_i,\varepsilon_i)+[A_{(1)}(P_i,\varepsilon_i)]^{\mu_1}\Big(-i\frac{\partial}{\partial P_n^{\mu_1}}\Big)+\cdots\Big]\prod_i^n e^{iP_i\cdot X} \tag{6.50}$$

In principle, one could use any external momenta to get the same result; however, this choice for n-th momenta is more convenient. Substituting $P_n=\sum_i^n P_i-\sum_j^{n-1}P_j$, and $P=\sum_i^n P_i$, we get the following identity,

$$\int d^{d+1}X\,(P\cdot X)\,e^{iP\cdot X} = -i\int d^{d+1}X\,X^{\mu}\frac{\partial}{\partial X^{\mu}}\,e^{iP\cdot X}$$
$$= i(d+1)\int d^{d+1}X\,e^{iP\cdot X} = i\,(d+1)\delta(P) \tag{6.51}$$

Here, the total derivative terms vanishes by choosing the appropriate boundary conditions. Using this identity, one can remove the ambiguity of X dependence from Eq.(6.50) and the momentum space n-point correlation thus can be written as,

$$\mathcal{A}(P_i) = \Big[A_{(0)}(P_i,\varepsilon_i) + [A_{(1)}(P_i,\varepsilon_i)]^{\mu_1}\Big(-i\frac{\partial}{\partial P_n^{\mu_1}}\Big) + \dots\Big]\delta\big(\sum_i P_i\big) \tag{6.52}$$

Following the arguments given in [236], we identify the leading order $\mathcal{S}$-Matrix $A_{(0)}$ as the "AdS $\mathcal{S}$-matrix". As has been explicitly shown in [236], this expression is related to the higher order $\mathcal{S}$-matrix terms via Ward identities or the conformal covariance of the correlation function $\mathcal{A}(P_i)$ under the action of the symmetry generators M_a. Hence, $A_{(0)}$ as the *AdS $\mathcal{S}$-matrix* is the only independent data needed to compute the momentum space correlator $\mathcal{A}(P_i)$ in the $1/R$ perturbation theory. In our work ([409]: in preparation), we have used this procedure to compute the four-point exchange diagram.

6.5 Conclusion

Building upon the formalism used in [236], we have studied the properties of conformally covariant momentum space representations of the CFT correlators, alternatively referred to as "AdS $\mathcal{S}$-Matrix" in the presence of massive vector fields. Under the flat space limit, where we keep the masses of the particles fixed under AdS radius $R \to \infty$, we have constructed the bulk-to-boundary propagator, and the bulk-to-bulk propagators for the massive vector fields perturbatively up to the sub-leading order in large R limit. To compute the n-point function of the currents in the boundary CFT, we need to compute the Witten diagrams [228], where the boundary of AdS is sketched as a circle, and the wavy lines are the gauge boson propagators.

To find the amplitude corresponding to these diagrams, we need to have the bulk vertices and the AdS propagators for the bulk fields. The simplest one of such diagrams is the contact diagram, which is an integrated diagram over a n-point vertex [410]. For massive vector fields, the bulk-to-bulk propagator is found in Eq.(6.44). Due to translational symmetry breaking in AdS, we must compute the Witten diagrams in the AdS momentum space for $1/R$ perturbation theory [236]. We expect the result to reduce back to the flat space Feynman diagrams computation in $R \to \infty$ limit. Our next goal is dedicated to finding the four-point Witten diagram in an effective theoretical model in AdS_{d+1} spacetime.

The results of our findings are consistent with the analysis for massive scalar fields done by the authors in [236]. In this direction, our work will help us to understand the properties of the *S-matrices* in curved spacetimes. To do this, we are required to compute all the n-point diagrams up to $\mathcal{O}(1/R^2)$. This can be done in principle using the propagators we have computed; however, that would be a tedious task. One way to simplify this would be to reconcile our analysis with the large gauge ward identity methods to deal with soft limits. A consequence of this would be the extension of the Infrared (IR) Triangle (which is well-known in flat space) to AdS field theories.

In the next chapter, I will conclude this thesis with the conclusions of and outlooks from each of the chapters separately and will highlight our biggest motivation for all these studies in line with holography.

CHAPTER 7

CONCLUSION

In this thesis, I put some recent updates on the modernized techniques used to understand flat space holography. Among many others, we were focused on the recently developed technique of celestial holography. Our focus is on finding the symmetries of asymptotically flat spacetimes. In Chapter 2, I illuminated the Celestial conformal field theoretic techniques, where I defined the *celestial* map at the level of wavefunctions, which helps us to map the scattering amplitude into the conformal fields correlators which transforms as primary fields under $SL(2,\mathbb{C})$ conformal symmetry group of the boundary CFT.

I stated the rules for finding the asymptotic symmetries and corresponding symmetry algebra in any theory using this technique, given the two most important ingredients, like the soft and collinear limits of the celestial scattering amplitudes. In certain theories, this technique turned out to be very helpful. Another motivation was the simplicity of the techniques, provided we have all the symmetries in theory. It is simpler than the usual killing vector approach. ence, computationally, it is desirable to develop a general algorithm for constructing the extensions of BMS algebra using the CCFT technique.

In the subsequent chapters of this thesis, I list the case studies that we did in the case of the bosonic Einstein-Yang-Mills theory and in supergravity theory. Along with this, we highlighted some use of the known double copy formalism as a requirement for our studies, which helps in computing quantum gravity amplitudes from gauge theory. Then, to conclude this thesis, I will light up our recent developments in understanding flat space from the perspective of an observer sitting in AdS spacetime itself.

Here is a summary of the conclusions from each chapter.

- In Chapter 3, we discussed the case of Einstein Yang-Mills theory, where our primary job was to construct the celestial conformal operators corresponding to the symmetry currents corresponding to the gravitons and gauge bosons in the theory. Due to the non-abelian nature of the symmetry generators, there was a demand to construct the *soft* limit independent composite current conformal operator. The shadow transformation helps us to construct the energy-momentum tensor (quasi-primary conformal operator of dimension $\Delta = 0$) of our CCFT from the *shadow* $\Delta = 2$ conformal primary operator. The

CCFT OPEs between the gauge boson operators with our defined energy-momentum tensor results in the correct Virasoro primaries. The structure of these celestial OPEs of the symmetry operators generating BMS put forth the relation to the extended BMS algebra $\mathfrak{bms}_4$ and in our case of EYM, the extension, $\mathfrak{eymbms}_4$:

$$\mathfrak{eymbms}_4 = \text{Superrotations} \uplus [\text{Supertranslations} \oplus \mathfrak{u}(N)\text{-gauge transformations}] \, .$$

Hence, the above algebraic statement showed our usual $\mathfrak{bms}_4$ with a symmetry extension of $\mathfrak{u}(N)$-gauge transformation. Similarly, for Einstein-Maxwell, we needed to switch off the color degrees of freedom to result in the extended algebra of $\mathfrak{embms}_4$ corresponding to $\mathfrak{u}(1)$ gauge-symmetry:

$$\mathfrak{embms}_4 = \text{Superrotations} \uplus [\text{Supertranslations} \oplus \mathfrak{u}(1)\text{-gauge transformations}] \, .$$

The upshot of this analysis is that Henneaux and Troessaert's methods of constructing the algebra give us the global Lorentz transformations; however, in our prescription, we have recovered the complete local superrotation algebra.

- In Chapter 4, we have made use of the famous formalism of double copy (DC), which relates a gravity scattering amplitude to gauge theory amplitude. However, we constructed the relation only in soft and collinear sectors of super Yang-Mills and $\mathcal{N} = 8$ supergravity theory. The important consideration in our work that helped in successful double copy relation is the unique self-duality condition. Based on the factorization of the states, we properly fixed the R-symmetry indices on both the gauge and gravity sides of the amplitudes.

  The goal of this work is twofold: first, we would like to construct the dual celestial CFT corresponding to the bulk $\mathcal{N} = 8$ supergravity in four dimensions. This requires the collinear limits of bulk amplitudes as they imply the OPEs of (super)conformal operators in the CCFT. Second, we would like to determine the asymptotic symmetries of $\mathcal{N} = 8$ supergravity using celestial holography. As discussed in [411], our final goal is to determine the contribution of BMS hairs to black hole entropy. The first step to such an analysis would be to understand the extension of the BMS group to *super* BMS group in $\mathcal{N} = 8$ supergravity. The corresponding $\mathcal{N} = 1$ supergravity case has already been worked out in [7]. Both these goals have been addressed in the work [392].

- In Chapter 5, CCFT techniques have been used to show the extension of the supersymmetry and non-extension of the R-symmetry algebra at the null infinities in our supergravity case study of $\mathcal{N} = 8$ theory.

  Let us briefly discuss the importance of the study of asymptotic symmetry algebra for higher $\mathcal{N}$ supergravity theories. Firstly, in the context of flat space holography[1], they can be used to find a field theory dual of the bulk theories. Supersymmetry gives us a more technical handle to address such questions. Secondly, it is conjectured [333, 416, 417] that the BMS hairs are responsible for Black Hole entropy. The conjecture has also been shown to be only partially correct in [418, 419], where the authors showed that the BMS hair could only partially incorporate some part of the Black Hole entropy. In the context of a class of black holes, namely extremal black holes in (super)string theories, the microscopic counting of black hole states is known in great detail. It would be interesting to find how much of this entropy is captured by the (super)$\mathfrak{bms}_4$ hairs. This project remains one of our prime goals to study in the future.

  It is instructive to note that our results are consistent with the usual expectation of symmetry enhancement at the boundary for gauge symmetries. In the case of ordinary gravity and minimal supergravity theories, the corresponding symmetries are local in nature, and hence, they have a natural infinite extension at the boundary. For extended supergravity, the R-symmetry is primarily a global symmetry, and in our study, we find that the symmetry group is not extended at the asymptotic boundary. In the CCFT language, this result comes from the regularity of the OPEs between the R-symmetry charges and the supersymmetry, which signifies the absence of collinear divergences. It would be nice to check the fate of R-symmetry in the context of gauged supergravity theories (where the R-symmetries are also local) by performing a direct asymptotic symmetry analysis of those theories. On the other hand, global non-compact symmetries, such as translation, also have a local counterpart in the theory of dynamical gravity, and hence, it does get an infinite extension at the asymptotic null boundary.

- In Chapter 6, we showcased some recent advancements in relation to the *scattering* in AdS spacetimes. One can object here to the obscurity in defining the scattering pro-

[1] In the context of 3-dimensional (super)gravity, the duals have been constructed in [337, 411–415].

cess, considering the closeness of the spacetime manifold. However, our analysis holds specific to the flat space limit of AdS spacetime. The $\mathcal{S}-$matrices of flat space can be derived by taking the dimension of the conformal field Δ in the CFT correlator to be large when the dual AdS length scale, $R \to \infty$. In the embedding space, the *AdS $\mathcal{S}$-matrix* is defined as the Fourier transform of the position space correlator that asserts complete information of the bulk physics in the conformal correlators in $1/R^2$ perturbations theory. We pursued the momentum Space prescription introduced in [236].

This chapter is attributed to our recent ongoing work [409]. The scattering, in our case, involves massive vector particles, and the results so far include the successful construction of the bulk-to-boundary propagator and the bulk-to-bulk propagators for the massive vector fields perturbatively up to the subleading order of $1/R^2$. This aims to explore Infrared sectors of 'Scattering amplitudes' in AdS spacetime. In addition, we want to relate it to more classical results or soft theorems in the $1/R$ perturbation theory. This momentum space analysis helps in the higher spin theories where the consistent formulation of the scattering is still missing.

Flat space IR behavior of massless particles becomes finite in AdS, which makes the AdS amplitudes more well-behaved than Minkowski spacetime to construct IR-safe observables by using AdS length to be the IR regulator. In the end, one can take this radius too large to get back to the flat space scattering rates and cross-sections. There are subtleties involved in studying massless particles in AdS and soft particles upon mass going zero limits. These are two different limits in AdS. The first one has been well explored in the literature [398, 406]; however, the notion of taking a soft limit is a bit obscure. This can be understood in the simultaneous double scaling limit, which is the soft limit in AdS. This was our prior motivation for studying scattering in AdS.

Again, the universal sectors of soft and collinear limits can be explored using the Mellin representations in the flat limit of AdS [420], unlike the momentum space formalism in our case. Hence, understanding this in the momentum space formalism could be one of the future aspects of this work. Another recently explored direction is to construct the Carrolian CFT Correlation functions from the AdS Witten diagrams, which is completely motivated by the $4D$ flat space celestial holography [195]; this could be a better exploration considering the mathematical simplicity of flat space holography and the relation of CCFT correlation function to Witten diagrams in the flat limit of AdS.

Finally, in this thesis, we presented some applications of asymptotic symmetry analysis, which relies on suitable *celestial* OPEs in the CCFT, connecting bulk and boundary physics, thus developing the holographic principle in flat spacetime. The case studies have been in the case of EYM and maximally supersymmetric $\mathcal{N} = 8$ Supergravity theory. This analysis strengthens our motivations to study this holography on or near the black hole horizons, contributing to the studies of *soft* hairs. Additionally, we are hopeful about understanding the flat space scattering from the perspective of the AdS observer, which will lead us to understand the universalities of the scattering amplitudes in the non-trivial background spacetimes. In our upcoming work, we will update the reader on this line of studies.

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BIBLIOGRAPHY

[1] Temple He et al. "New Symmetries of Massless QED". In: *JHEP* 10 (2014), p. 112. DOI: 10.1007/JHEP10(2014)112. arXiv: 1407.3789 [hep-th].

[2] Temple He, Prahar Mitra, and Andrew Strominger. "2D Kac-Moody Symmetry of 4D Yang-Mills Theory". In: *JHEP* 10 (2016), p. 137. DOI: 10.1007/JHEP10(2016)137. arXiv: 1503.02663 [hep-th].

[3] Thomas T. Dumitrescu et al. "Infinite-dimensional fermionic symmetry in supersymmetric gauge theories". In: *JHEP* 08 (2021), p. 051. DOI: 10.1007/JHEP08(2021)051. arXiv: 1511.07429 [hep-th].

[4] Glenn Barnich and Cédric Troessaert. "Symmetries of Asymptotically Flat Four-Dimensional Spacetimes at Null Infinity Revisited". In: *Phys. Rev. Lett.* 105 (11 Sept. 2010), p. 111103. DOI: 10.1103/PhysRevLett.105.111103. URL: https://link.aps.org/doi/10.1103/PhysRevLett.105.111103.

[5] R. Sachs. "Asymptotic Symmetries in Gravitational Theory". In: *Phys. Rev.* 128 (6 Dec. 1962), pp. 2851–2864. DOI: 10.1103/PhysRev.128.2851. URL: https://link.aps.org/doi/10.1103/PhysRev.128.2851.

[6] Andrew Strominger. "Asymptotic Symmetries of Yang-Mills Theory". In: *JHEP* 07 (2014), p. 151. DOI: 10.1007/JHEP07(2014)151. arXiv: 1308.0589 [hep-th].

[7] Angelos Fotopoulos et al. "Extended Super BMS Algebra of Celestial CFT". In: *JHEP* 09 (2020), p. 198. DOI: 10.1007/JHEP09(2020)198. arXiv: 2007.03785 [hep-th].

[8] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems". In: *Proc. Roy. Soc. Lond. A* 269 (1962), pp. 21–52. DOI: 10.1098/rspa.1962.0161.

[9] R. K. Sachs. "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times". In: *Proc. Roy. Soc. Lond. A* 270 (1962), pp. 103–126. DOI: 10.1098/rspa.1962.0206.

[10] R. Sachs. "Asymptotic symmetries in gravitational theory". In: *Phys. Rev.* 128 (1962), pp. 2851–2864. DOI: 10.1103/PhysRev.128.2851.

[11] Andrew Strominger. "Lectures on the Infrared Structure of Gravity and Gauge Theory". In: (Mar. 2017). arXiv: 1703.05448 [hep-th].

[12] Temple He et al. "BMS supertranslations and Weinberg's soft graviton theorem". In: *JHEP* 05 (2015), p. 151. DOI: 10.1007/JHEP05(2015)151. arXiv: 1401.7026 [hep-th].

[13] Glenn Barnich and Cedric Troessaert. "BMS charge algebra". In: *JHEP* 12 (2011), p. 105. DOI: 10.1007/JHEP12(2011)105. arXiv: 1106.0213 [hep-th].

[14] Glenn Barnich and Cédric Troessaert. "Comments on holographic current algebras and asymptotically flat four dimensional spacetimes at null infinity". In: *JHEP* 11 (2013), p. 003. DOI: 10.1007/JHEP11(2013)003. arXiv: 1309.0794 [hep-th].

[15] Andrew Strominger. "On BMS Invariance of Gravitational Scattering". In: *JHEP* 07 (2014), p. 152. DOI: 10.1007/JHEP07(2014)152. arXiv: 1312.2229 [hep-th].

[16] Barak Gabai and Amit Sever. "Large gauge symmetries and asymptotic states in QED". In: *JHEP* 12 (2016), p. 095. DOI: 10.1007/JHEP12(2016)095. arXiv: 1607.08599 [hep-th].

[17] Marc Favata. “The gravitational-wave memory effect”. In: *Class. Quant. Grav.* 27 (2010). Ed. by Zsuzsa Marka and Szabolcs Marka, p. 084036. DOI: 10.1088/0264-9381/27/8/084036. arXiv: 1003.3486 [gr-qc].

[18] Demetrios Christodoulou. “Nonlinear nature of gravitation and gravitational-wave experiments”. In: *Phys. Rev. Lett.* 67 (12 Sept. 1991), pp. 1486–1489. DOI: 10.1103/PhysRevLett.67.1486. URL: https://link.aps.org/doi/10.1103/PhysRevLett.67.1486.

[19] Y. B. Zel’dovich and A. G. Polnarev. “Radiation of gravitational waves by a cluster of superdense stars”. In: *Sov. Astron.* 18 (1974), p. 17.

[20] Leonard Susskind. “Electromagnetic Memory”. In: (July 2015). arXiv: 1507.02584 [hep-th].

[21] Sabrina Pasterski. “Asymptotic Symmetries and Electromagnetic Memory”. In: *JHEP* 09 (2017), p. 154. DOI: 10.1007/JHEP09(2017)154. arXiv: 1505.00716 [hep-th].

[22] Steven Carlip. “The (2+1)-Dimensional black hole”. In: *Class. Quant. Grav.* 12 (1995), pp. 2853–2880. DOI: 10.1088/0264-9381/12/12/005. arXiv: gr-qc/9506079.

[23] Hernando Quevedo and Alberto Sanchez. “Geometric description of BTZ black holes thermodynamics”. In: *Phys. Rev. D* 79 (2009), p. 024012. DOI: 10.1103/PhysRevD.79.024012. arXiv: 0811.2524 [gr-qc].

[24] Maximo Banados, Claudio Teitelboim, and Jorge Zanelli. “The Black hole in three-dimensional space-time”. In: *Phys. Rev. Lett.* 69 (1992), pp. 1849–1851. DOI: 10.1103/PhysRevLett.69.1849. arXiv: hep-th/9204099.

[25] Gábor Sárosi. “AdS_2 holography and the SYK model”. In: *PoS* Modave2017 (2018), p. 001. DOI: 10.22323/1.323.0001. arXiv: 1711.08482 [hep-th].

[26] Hamid Afshar et al. “Flat space holography and the complex Sachdev-Ye-Kitaev model”. In: *Phys. Rev. D* 101.8 (2020), p. 086024. DOI: 10.1103/PhysRevD.101.086024. arXiv: 1911.05739 [hep-th].

[27] Phil Saad, Stephen H. Shenker, and Douglas Stanford. “JT gravity as a matrix integral”. In: (Mar. 2019). arXiv: 1903.11115 [hep-th].

[28] Nathan Benjamin, Scott Collier, and Alexander Maloney. “Pure Gravity and Conical Defects”. In: *JHEP* 09 (2020), p. 034. DOI: 10.1007/JHEP09(2020)034. arXiv: 2004.14428 [hep-th].

[29] M. Kenmoku, S. Uchida, and T. Matsuyama. “Conical singular solutions in (2+1)-dimensional gravity employing the ADM canonical formalism”. In: *Int. J. Mod. Phys. D* 12 (2003), pp. 677–687. DOI: 10.1142/S021827180300313X. arXiv: gr-qc/0206049.

[30] G Clement. “Classical solutions in three-dimensional Einstein-Maxwell cosmological gravity”. In: *Classical and Quantum Gravity* 10.5 (May 1993), p. L49. DOI: 10.1088/0264-9381/10/5/002. URL: https://dx.doi.org/10.1088/0264-9381/10/5/002.

[31] Roger Penrose. “Asymptotic Properties of Fields and Space-Times”. In: *Phys. Rev. Lett.* 10 (2 Jan. 1963), pp. 66–68. DOI: 10.1103/PhysRevLett.10.66. URL: https://link.aps.org/doi/10.1103/PhysRevLett.10.66.

[32] Stephen W. Hawking and George F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Feb. 2023. ISBN: 978-1-009-25316-1, 978-1-009-25315-4, 978-0-521-20016-5, 978-0-521-09906-6, 978-0-511-82630-6, 978-0-521-09906-6. DOI: 10.1017/9781009253161.

[33] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. San Francisco: Addison-Wesley, 2004. ISBN: 978-0-8053-8732-2.

[34] Ray d'Inverno. *Introducing Einstein's Relativity*. Oxford: Oxford University Press, 1992. ISBN: 978-0-19-859686-8.

[35] Geoffrey Compère. *Advanced Lectures on General Relativity*. Cham: Springer, 2019. ISBN: 978-3-030-04260-8. DOI: 10.1007/978-3-030-04260-8.

[36] Ezra T. Newman. "Light-Cones, Almost Light-Cones and Almost-Complex Light-Cones". In: *Gen. Rel. Grav.* 49.8 (2017), p. 102. DOI: 10.1007/s10714-017-2264-1. arXiv: 1704.04971 [gr-qc].

[37] Laura Donnay. "Celestial holography: An asymptotic symmetry perspective". In: *Phys. Rept.* 1073 (2024), pp. 1–41. DOI: 10.1016/j.physrep.2024.04.003. arXiv: 2310.12922 [hep-th].

[38] D. Christodoulou and S. Klainerman. "The Global nonlinear stability of the Minkowski space". In: (1993).

[39] Sabrina Pasterski, Andrea Puhm, and Emilio Trevisani. "Revisiting the conformally soft sector with celestial diamonds". In: *JHEP* 11 (2021), p. 143. DOI: 10.1007/JHEP11(2021)143. arXiv: 2105.09792 [hep-th].

[40] David Poland, Slava Rychkov, and Alessandro Vichi. "The conformal bootstrap: Theory, numerical techniques, and applications". In: *Rev. Mod. Phys.* 91 (1 Jan. 2019), p. 015002. DOI: 10.1103/RevModPhys.91.015002. URL: https://link.aps.org/doi/10.1103/RevModPhys.91.015002.

[41] David Poland and David Simmons-Duffin. "The conformal bootstrap". In: *Nature Phys.* 12.6 (2016), pp. 535–539. DOI: 10.1038/nphys3761.

[42] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. "Infinite conformal symmetry in two-dimensional quantum field theory". In: *Nuclear Physics B* 241.2 (1984), pp. 333–380. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(84)90052-X. URL: https://www.sciencedirect.com/science/article/pii/055032138490052X.

[43] Arnab Priya Saha, Biswajit Sahoo, and Ashoke Sen. "Proof of the classical soft graviton theorem in $D = 4$". In: *JHEP* 06 (2020), p. 153. DOI: 10.1007/JHEP06(2020)153. arXiv: 1912.06413 [hep-th].

[44] Steven Weinberg. "Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass". In: *Phys. Rev.* 135 (4B Aug. 1964), B1049–B1056. DOI: 10.1103/PhysRev.135.B1049. URL: https://link.aps.org/doi/10.1103/PhysRev.135.B1049.

[45] Steven Weinberg. "Infrared Photons and Gravitons". In: *Phys. Rev.* 140 (2B Oct. 1965), B516–B524. DOI: 10.1103/PhysRev.140.B516. URL: https://link.aps.org/doi/10.1103/PhysRev.140.B516.

[46] David J. Gross and Roman Jackiw. "Low-Energy Theorem for Graviton Scattering". In: *Phys. Rev.* 166 (5 Feb. 1968), pp. 1287–1292. DOI: 10.1103/PhysRev.166.1287. URL: https://link.aps.org/doi/10.1103/PhysRev.166.1287.

[47] Miguel Campiglia and Alok Laddha. "Loop Corrected Soft Photon Theorem as a Ward Identity". In: *JHEP* 10 (2019), p. 287. DOI: 10.1007/JHEP10(2019)287. arXiv: 1903.09133 [hep-th].

[48] Biswajit Sahoo. “Classical Sub-subleading Soft Photon and Soft Graviton Theorems in Four Spacetime Dimensions”. In: *JHEP* 12 (2020), p. 070. DOI: 10.1007/JHEP12(2020)070. arXiv: 2008.04376 [hep-th].

[49] Sayali Atul Bhatkar. “Ward identity for loop level soft photon theorem for massless QED coupled to gravity”. In: *JHEP* 10 (2020), p. 110. DOI: 10.1007/JHEP10(2020)110. arXiv: 1912.10229 [hep-th].

[50] Eliot Hijano and Dominik Neuenfeld. “Soft photon theorems from CFT Ward identites in the flat limit of AdS/CFT”. In: *JHEP* 11 (2020), p. 009. DOI: 10.1007/JHEP11(2020)009. arXiv: 2005.03667 [hep-th].

[51] Pujian Mao and Jun-Bao Wu. “Note on asymptotic symmetries and soft gluon theorems”. In: *Phys. Rev. D* 96.6 (2017), p. 065023. DOI: 10.1103/PhysRevD.96.065023. arXiv: 1704.05740 [hep-th].

[52] Stefano Catani and Massimiliano Grazzini. “The soft gluon current at one loop order”. In: *Nucl. Phys. B* 591 (2000), pp. 435–454. DOI: 10.1016/S0550-3213(00)00572-1. arXiv: hep-ph/0007142.

[53] F. E. Low. “Bremsstrahlung of Very Low-Energy Quanta in Elementary Particle Collisions”. In: *Phys. Rev.* 110 (4 May 1958), pp. 974–977. DOI: 10.1103/PhysRev.110.974. URL: https://link.aps.org/doi/10.1103/PhysRev.110.974.

[54] T. H. Burnett and Norman M. Kroll. “Extension of the Low Soft-Photon Theorem”. In: *Phys. Rev. Lett.* 20 (2 Jan. 1968), pp. 86–88. DOI: 10.1103/PhysRevLett.20.86. URL: https://link.aps.org/doi/10.1103/PhysRevLett.20.86.

[55] Eduardo Casali. “Soft sub-leading divergences in Yang-Mills amplitudes”. In: *JHEP* 08 (2014), p. 077. DOI: 10.1007/JHEP08(2014)077. arXiv: 1404.5551 [hep-th].

[56] Temple He and Prahar Mitra. “Asymptotic symmetries and Weinberg’s soft photon theorem in Mink_{d+2}”. In: *JHEP* 10 (2019), p. 213. DOI: 10.1007/JHEP10(2019)213. arXiv: 1903.02608 [hep-th].

[57] Freddy Cachazo and Andrew Strominger. “Evidence for a New Soft Graviton Theorem”. In: (Apr. 2014). arXiv: 1404.4091 [hep-th].

[58] P. Di Francesco, P. Mathieu, and D. Sénéchal. *Conformal Field Theory*. Graduate texts in contemporary physics. Island Press, 1996. ISBN: 9781461222576. URL: https://books.google.com/books?id=mcMbswEACAAJ.

[59] Arjun Bagchi. “Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories”. In: *Phys. Rev. Lett.* 105 (2010), p. 171601. DOI: 10.1103/PhysRevLett.105.171601. arXiv: 1006.3354 [hep-th].

[60] Duccio Pappadopulo et al. “OPE Convergence in Conformal Field Theory”. In: *Phys. Rev. D* 86 (2012), p. 105043. DOI: 10.1103/PhysRevD.86.105043. arXiv: 1208.6449 [hep-th].

[61] Wei Fan et al. “Elements of celestial conformal field theory”. In: *JHEP* 08 (2022), p. 213. DOI: 10.1007/JHEP08(2022)213. arXiv: 2202.08288 [hep-th].

[62] Steven G. Avery and Burkhard U. W. Schwab. “Noether’s second theorem and Ward identities for gauge symmetries”. In: *JHEP* 02 (2016), p. 031. DOI: 10.1007/JHEP02(2016)031. arXiv: 1510.07038 [hep-th].

[63] L.F. Abbott and S. Deser. “Charge definition in non-abelian gauge theories”. In: *Physics Letters B* 116.4 (1982), pp. 259–263. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(82)90338-0. URL: https://www.sciencedirect.com/science/article/pii/0370269382903380.

[64] L.F. Abbott and S. Deser. “Stability of gravity with a cosmological constant”. In: *Nuclear Physics B* 195.1 (1982), pp. 76–96. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(82)90049-9. URL: https://www.sciencedirect.com/science/article/pii/0550321382900499.

[65] G.’t Hooft. “Magnetic monopoles in unified gauge theories”. In: *Nuclear Physics B* 79.2 (1974), pp. 276–284. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(74)90486-6. URL: https://www.sciencedirect.com/science/article/pii/0550321374904866.

[66] B. Julia and A. Zee. “Poles with both magnetic and electric charges in non-Abelian gauge theory”. In: *Phys. Rev. D* 11 (8 Apr. 1975), pp. 2227–2232. DOI: 10.1103/PhysRevD.11.2227. URL: https://link.aps.org/doi/10.1103/PhysRevD.11.2227.

[67] John M. Cornwall and George Tiktopoulos. “Infrared behavior of non-Abelian gauge theories”. In: *Phys. Rev. D* 13 (12 June 1976), pp. 3370–3397. DOI: 10.1103/PhysRevD.13.3370. URL: https://link.aps.org/doi/10.1103/PhysRevD.13.3370.

[68] David J. Gross and Frank Wilczek. “Asymptotically Free Gauge Theories. I”. In: *Phys. Rev. D* 8 (10 Nov. 1973), pp. 3633–3652. DOI: 10.1103/PhysRevD.8.3633. URL: https://link.aps.org/doi/10.1103/PhysRevD.8.3633.

[69] John M. Cornwall, David N. Levin, and George Tiktopoulos. “Derivation of gauge invariance from high-energy unitarity bounds on the *S* matrix”. In: *Phys. Rev. D* 10 (4 Aug. 1974), pp. 1145–1167. DOI: 10.1103/PhysRevD.10.1145. URL: https://link.aps.org/doi/10.1103/PhysRevD.10.1145.

[70] Riccardo Gonzo et al. “Asymptotic charges and coherent states in QCD”. In: *Phys. Rev. D* 104 (2 July 2021), p. 025019. DOI: 10.1103/PhysRevD.104.025019. URL: https://link.aps.org/doi/10.1103/PhysRevD.104.025019.

[71] Riccardo Gonzo et al. “Asymptotic charges and coherent states in QCD”. In: *Phys. Rev. D* 104.2 (2021), p. 025019. DOI: 10.1103/PhysRevD.104.025019. arXiv: 1906.11763 [hep-th].

[72] Glenn Barnich and Friedemann Brandt. “Covariant theory of asymptotic symmetries, conservation laws and central charges”. In: *Nucl. Phys. B* 633 (2002), pp. 3–82. DOI: 10.1016/S0550-3213(02)00251-1. arXiv: hep-th/0111246.

[73] Glenn Barnich. “Boundary charges in gauge theories: Using Stokes theorem in the bulk”. In: *Class. Quant. Grav.* 20 (2003), pp. 3685–3698. DOI: 10.1088/0264-9381/20/16/310. arXiv: hep-th/0301039.

[74] Anjalika Nande, Monica Pate, and Andrew Strominger. “Soft Factorization in QED from 2D Kac-Moody Symmetry”. In: *JHEP* 02 (2018), p. 079. DOI: 10.1007/JHEP02(2018)079. arXiv: 1705.00608 [hep-th].

[75] Tom Banks and Willy Fischler. “A Model for high-energy scattering in quantum gravity”. In: (June 1999). arXiv: hep-th/9906038.

[76] Herman L. Verlinde and Erik P. Verlinde. “Scattering at Planckian energies”. In: *Nucl. Phys. B* 371 (1992), pp. 246–268. DOI: 10.1016/0550-3213(92)90236-5. arXiv: hep-th/9110017.

[77] D. Amati, M. Ciafaloni, and G. Veneziano. “Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions”. In: *Int. J. Mod. Phys. A* 3 (1988), pp. 1615–1661. DOI: 10.1142/S0217751X88000710.

[78] D. Amati, M. Ciafaloni, and G. Veneziano. “Superstring collisions at planckian energies”. In: *Physics Letters B* 197.1 (1987), pp. 81–88. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(87)90346-7. URL: https://www.sciencedirect.com/science/article/pii/0370269387903467.

[79] I. J. Muzinich and M. Soldate. “High-energy unitarity of gravitation and strings”. In: *Phys. Rev. D* 37 (2 Jan. 1988), pp. 359–367. DOI: 10.1103/PhysRevD.37.359. URL: https://link.aps.org/doi/10.1103/PhysRevD.37.359.

[80] Gerard ’t Hooft. “Graviton Dominance in Ultrahigh-Energy Scattering”. In: *Phys. Lett. B* 198 (1987), pp. 61–63. DOI: 10.1016/0370-2693(87)90159-6.

[81] D. Amati, M. Ciafaloni, and G. Veneziano. “Higher-order gravitational deflection and soft bremsstrahlung in planckian energy superstring collisions”. In: *Nuclear Physics B* 347.3 (1990), pp. 550–580. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(90)90375-N. URL: https://www.sciencedirect.com/science/article/pii/055032139090375N.

[82] B. P. Abbott et al. “Observation of Gravitational Waves from a Binary Black Hole Merger”. In: *Phys. Rev. Lett.* 116 (6 Feb. 2016), p. 061102. DOI: 10.1103/PhysRevLett.116.061102. URL: https://link.aps.org/doi/10.1103/PhysRevLett.116.061102.

[83] B. P. Abbott et al. “Prospects for observing and localizing gravitational-wave transients with Advanced LIGO, Advanced Virgo and KAGRA”. In: *Living Rev. Rel.* 19 (2016), p. 1. DOI: 10.1007/s41114-020-00026-9. arXiv: 1304.0670 [gr-qc].

[84] C. S. Unnikrishnan. “IndIGO and LIGO-India: Scope and plans for gravitational wave research and precision metrology in India”. In: *Int. J. Mod. Phys. D* 22 (2013), p. 1341010. DOI: 10.1142/S0218271813410101. arXiv: 1510.06059 [physics.ins-det].

[85] Bala Iyer. “Ligo-india”. In: ().

[86] Yoichi Iwasaki. “Quantum Theory of Gravitation vs. Classical Theory*): Fourth-Order Potential”. In: *Progress of Theoretical Physics* 46.5 (Nov. 1971), pp. 1587–1609. ISSN: 0033-068X. DOI: 10.1143/PTP.46.1587. eprint: https://academic.oup.com/ptp/article-pdf/46/5/1587/5271183/46-5-1587.pdf. URL: https://doi.org/10.1143/PTP.46.1587.

[87] Zvi Bern et al. “Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order”. In: *Phys. Rev. Lett.* 122 (20 May 2019), p. 201603. DOI: 10.1103/PhysRevLett.122.201603. URL: https://link.aps.org/doi/10.1103/PhysRevLett.122.201603.

[88] Zvi Bern et al. “Black Hole Binary Dynamics from the Double Copy and Effective Theory”. In: *JHEP* 10 (2019), p. 206. DOI: 10.1007/JHEP10(2019)206. arXiv: 1908.01493 [hep-th].

[89] Zvi Bern et al. “One-loop n-point gauge theory amplitudes, unitarity and collinear limits”. In: *Nuclear Physics B* 425.1 (1994), pp. 217–260. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(94)90179-1. URL: https://www.sciencedirect.com/science/article/pii/0550321394901791.

[90] Z. Bern et al. “Maximally supersymmetric planar Yang-Mills amplitudes at five loops”. In: *Phys. Rev. D* 76 (12 Dec. 2007), p. 125020. DOI: 10.1103/PhysRevD.76.125020. URL: https://link.aps.org/doi/10.1103/PhysRevD.76.125020.

[91] “A relation between tree amplitudes of closed and open strings”. In: *Nuclear Physics B* 269.1 (1986), pp. 1–23.

[92] Thibault Damour. “High-energy gravitational scattering and the general relativistic two-body problem”. In: *Phys. Rev. D* 97 (4 Feb. 2018), p. 044038. DOI: 10.1103/PhysRevD.97.044038. URL: https://link.aps.org/doi/10.1103/PhysRevD.97.044038.

[93] A. Buonanno and T. Damour. “Effective one-body approach to general relativistic two-body dynamics”. In: *Phys. Rev. D* 59 (8 Mar. 1999), p. 084006. DOI: 10.1103/PhysRevD.59.084006. URL: https://link.aps.org/doi/10.1103/PhysRevD.59.084006.

[94] Thibault Damour. “Classical and quantum scattering in post-Minkowskian gravity”. In: *Phys. Rev. D* 102.2 (2020), p. 024060. DOI: 10.1103/PhysRevD.102.024060. arXiv: 1912.02139 [gr-qc].

[95] Duff Neill and Ira Z. Rothstein. “Classical Space-Times from the S Matrix”. In: *Nucl. Phys. B* 877 (2013), pp. 177–189. DOI: 10.1016/j.nuclphysb.2013.09.007. arXiv: 1304.7263 [hep-th].

[96] Ruth Britto et al. “Direct proof of tree-level recursion relation in Yang-Mills theory”. In: *Phys. Rev. Lett.* 94 (2005), p. 181602. DOI: 10.1103/PhysRevLett.94.181602. arXiv: hep-th/0501052.

[97] Zvi Bern et al. “Fusing gauge theory tree amplitudes into loop amplitudes”. In: *Nucl. Phys. B* 435 (1995), pp. 59–101. DOI: 10.1016/0550-3213(94)00488-Z. arXiv: hep-ph/9409265.

[98] Gianluca Delfino, Kirill Krasnov, and Carlos Scarinci. “Pure connection formalism for gravity: Recursion relations”. In: *JHEP* 03 (2015), p. 120. DOI: 10.1007/JHEP03(2015)120. arXiv: 1410.5647 [hep-th].

[99] Carlos R. Mafra and Oliver Schlotterer. “Berends-Giele recursions and the BCJ duality in superspace and components”. In: *JHEP* 03 (2016), p. 097. DOI: 10.1007/JHEP03(2016)097. arXiv: 1510.08846 [hep-th].

[100] Kanghoon Lee. “Quantum off-shell recursion relation”. In: *JHEP* 05 (2022), p. 051. DOI: 10.1007/JHEP05(2022)051. arXiv: 2202.08133 [hep-th].

[101] Kyoungho Cho, Kwangeon Kim, and Kanghoon Lee. “The off-shell recursion for gravity and the classical double copy for currents”. In: *JHEP* 01 (2022), p. 186. DOI: 10.1007/JHEP01(2022)186. arXiv: 2109.06392 [hep-th].

[102] Simon Badger et al. “Comparing efficient computation methods for massless QCD tree amplitudes: Closed analytic formulas versus Berends-Giele recursion”. In: *Phys. Rev. D* 87 (3 Feb. 2013), p. 034011. DOI: 10.1103/PhysRevD.87.034011. URL: https://link.aps.org/doi/10.1103/PhysRevD.87.034011.

[103] F.A. Berends and W.T. Giele. “Recursive calculations for processes with n gluons”. In: *Nuclear Physics B* 306.4 (1988), pp. 759–808. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(88)90442-7. URL: https://www.sciencedirect.com/science/article/pii/0550321388904427.

[104] Zvi Bern et al. "One loop n point gauge theory amplitudes, unitarity and collinear limits". In: *Nucl. Phys. B* 425 (1994), pp. 217–260. DOI: 10.1016/0550-3213(94)90179-1. arXiv: hep-ph/9403226.

[105] Zvi Bern, Lance J. Dixon, and Radu Roiban. "Is N = 8 supergravity ultraviolet finite?" In: *Phys. Lett. B* 644 (2007), pp. 265–271. DOI: 10.1016/j.physletb.2006.11.030. arXiv: hep-th/0611086.

[106] Nima Arkani-Hamed, Freddy Cachazo, and Jared Kaplan. "What is the Simplest Quantum Field Theory?" In: *JHEP* 09 (2010), p. 016. DOI: 10.1007/JHEP09(2010)016. arXiv: 0808.1446 [hep-th].

[107] F. Bloch and A. Nordsieck. "Note on the Radiation Field of the Electron". In: *Phys. Rev.* 52 (2 July 1937), pp. 54–59. DOI: 10.1103/PhysRev.52.54. URL: https://link.aps.org/doi/10.1103/PhysRev.52.54.

[108] Toichiro Kinoshita. "Mass Singularities of Feynman Amplitudes". In: *Journal of Mathematical Physics* 3.4 (July 1962), pp. 650–677. ISSN: 0022-2488. DOI: 10.1063/1.1724268. eprint: https://pubs.aip.org/aip/jmp/article-pdf/3/4/650/19167464/650_1_online.pdf. URL: https://doi.org/10.1063/1.1724268.

[109] T. D. Lee and M. Nauenberg. "Degenerate Systems and Mass Singularities". In: *Phys. Rev.* 133 (6B Mar. 1964), B1549–B1562. DOI: 10.1103/PhysRev.133.B1549. URL: https://link.aps.org/doi/10.1103/PhysRev.133.B1549.

[110] Julio Parra-Martinez, Michael S. Ruf, and Mao Zeng. "Extremal black hole scattering at $\mathcal{O}(G^3)$: graviton dominance, eikonal exponentiation, and differential equations". In: *JHEP* 11 (2020), p. 023. DOI: 10.1007/JHEP11(2020)023. arXiv: 2005.04236 [hep-th].

[111] G.'t Hooft. "Graviton dominance in ultra-high-energy scattering". In: *Physics Letters B* 198.1 (1987), pp. 61–63. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(87)90159-6. URL: https://www.sciencedirect.com/science/article/pii/0370269387901596.

[112] Paolo Di Vecchia et al. "Universality of ultra-relativistic gravitational scattering". In: *Phys. Lett. B* 811 (2020), p. 135924. DOI: 10.1016/j.physletb.2020.135924. arXiv: 2008.12743 [hep-th].

[113] Song He, Yu-tin Huang, and Congkao Wen. "Loop Corrections to Soft Theorems in Gauge Theories and Gravity". In: *JHEP* 12 (2014), p. 115. DOI: 10.1007/JHEP12(2014)115. arXiv: 1405.1410 [hep-th].

[114] Zvi Bern, Scott Davies, and Josh Nohle. "On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons". In: *Phys. Rev. D* 90.8 (2014), p. 085015. DOI: 10.1103/PhysRevD.90.085015. arXiv: 1405.1015 [hep-th].

[115] Anupam A. H, Arpan Kundu, and Krishnendu Ray. "Double soft graviton theorems and Bondi-Metzner-Sachs symmetries". In: *Phys. Rev. D* 97 (10 May 2018), p. 106019. DOI: 10.1103/PhysRevD.97.106019. URL: https://link.aps.org/doi/10.1103/PhysRevD.97.106019.

[116] Yuta Hamada and Gary Shiu. "Infinite Set of Soft Theorems in Gauge-Gravity Theories as Ward-Takahashi Identities". In: *Phys. Rev. Lett.* 120 (20 May 2018), p. 201601. DOI: 10.1103/PhysRevLett.120.201601. URL: https://link.aps.org/doi/10.1103/PhysRevLett.120.201601.

[117] Alok Laddha and Ashoke Sen. "Sub-subleading Soft Graviton Theorem in Generic Theories of Quantum Gravity". In: *JHEP* 10 (2017), p. 065. DOI: 10.1007/JHEP10(2017)065. arXiv: 1706.00759 [hep-th].

[118] Thomas Klose et al. "Double-Soft Limits of Gluons and Gravitons". In: *JHEP* 07 (2015), p. 135. DOI: 10.1007/JHEP07(2015)135. arXiv: 1504.05558 [hep-th].

[119] R. Jackiw. "Low-Energy Theorems for Massless Bosons: Photons and Gravitons". In: *Phys. Rev.* 168 (5 Apr. 1968), pp. 1623–1633. DOI: 10.1103/PhysRev.168.1623. URL: https://link.aps.org/doi/10.1103/PhysRev.168.1623.

[120] Zvi Bern et al. "Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance". In: *Phys. Rev. D* 90.8 (2014), p. 084035. DOI: 10.1103/PhysRevD.90.084035. arXiv: 1406.6987 [hep-th].

[121] Miguel Campiglia and Alok Laddha. "Asymptotic symmetries and subleading soft graviton theorem". In: *Phys. Rev. D* 90.12 (2014), p. 124028. DOI: 10.1103/PhysRevD.90.124028. arXiv: 1408.2228 [hep-th].

[122] Tomasz R. Taylor. "A Course in Amplitudes". In: *Phys. Rept.* 691 (2017), pp. 1–37. DOI: 10.1016/j.physrep.2017.05.002. arXiv: 1703.05670 [hep-th].

[123] Z. Bern et al. "Multi-leg one-loop gravity amplitudes from gauge theory". In: *Nuclear Physics B* 546.1 (1999), pp. 423–479. ISSN: 0550-3213. DOI: https://doi.org/10.1016/S0550-3213(99)00029-2. URL: https://www.sciencedirect.com/science/article/pii/S0550321399000292.

[124] Zvi Bern, Vittorio Del Duca, and Carl R. Schmidt. "The infrared behavior of one-loop gluon amplitudes at next-to-next-to-leading order". In: *Physics Letters B* 445.1 (1998), pp. 168–177. ISSN: 0370-2693. DOI: https://doi.org/10.1016/S0370-2693(98)01495-6. URL: https://www.sciencedirect.com/science/article/pii/S0370269398014956.

[125] Michelangelo L. Mangano and Stephen J. Parke. "Multi-parton amplitudes in gauge theories". In: *Physics Reports* 200.6 (1991), pp. 301–367. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(91)90091-Y. URL: https://www.sciencedirect.com/science/article/pii/037015739190091Y.

[126] Livia Ferro, Tomasz Łukowski, and Robert Moerman. "From momentum amplituhedron boundaries toamplitude singularities and back". In: *JHEP* 07.07 (2020), p. 201. DOI: 10.1007/JHEP07(2020)201. arXiv: 2003.13704 [hep-th].

[127] S. Carlip. "Black Hole Entropy from Bondi-Metzner-Sachs Symmetry at the Horizon". In: *Phys. Rev. Lett.* 120.10 (2018), p. 101301. DOI: 10.1103/PhysRevLett.120.101301. arXiv: 1702.04439 [gr-qc].

[128] Ashoke Sen. "Black Hole Entropy Function, Attractors and Precision Counting of Microstates". In: *Gen. Rel. Grav.* 40 (2008), pp. 2249–2431. DOI: 10.1007/s10714-008-0626-4. arXiv: 0708.1270 [hep-th].

[129] Ipsita Mandal and Ashoke Sen. "Black Hole Microstate Counting and its Macroscopic Counterpart". In: *Class. Quant. Grav.* 27 (2010). Ed. by Laurent Baulieu et al., p. 214003. DOI: 10.1088/0264-9381/27/21/214003. arXiv: 1008.3801 [hep-th].

[130] Atish Dabholkar. "Exact counting of black hole microstates". In: *Phys. Rev. Lett.* 94 (2005), p. 241301. DOI: 10.1103/PhysRevLett.94.241301. arXiv: hep-th/0409148.

[131] Steven Carlip. “Effective Conformal Descriptions of Black Hole Entropy: A Review”. In: *AIP Conf. Proc.* 1483.1 (2012). Ed. by Waldyr Alves Rodrigues et al., pp. 54–62. DOI: 10.1063/1.4756962. arXiv: 1207.1488 [gr-qc].

[132] Andrew Strominger and Alexander Zhiboedov. “Gravitational Memory, BMS Supertranslations and Soft Theorems”. In: *JHEP* 01 (2016), p. 086. DOI: 10.1007/JHEP01(2016)086. arXiv: 1411.5745 [hep-th].

[133] Glenn Barnich and Cedric Troessaert. “Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited”. In: *Phys. Rev. Lett.* 105 (2010), p. 111103. DOI: 10.1103/PhysRevLett.105.111103. arXiv: 0909.2617 [gr-qc].

[134] Glenn Barnich and Cedric Troessaert. “Aspects of the BMS/CFT correspondence”. In: *JHEP* 05 (2010), p. 062. DOI: 10.1007/JHEP05(2010)062. arXiv: 1001.1541 [hep-th].

[135] T. Banks. “A Critique of pure string theory: Heterodox opinions of diverse dimensions”. In: (June 2003). arXiv: hep-th/0306074.

[136] A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. “Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory”. In: *Nucl. Phys. B* 241 (1984). Ed. by I. M. Khalatnikov and V. P. Mineev, pp. 333–380. DOI: 10.1016/0550-3213(84)90052-X.

[137] Steven Weinberg. *The quantum theory of fields. Vol. 3: Supersymmetry*. Cambridge University Press, June 2013. ISBN: 978-0-521-67055-5, 978-1-139-63263-8, 978-0-521-67055-5.

[138] M.A. Awada, G.W. Gibbons, and W.T. Shaw. “Conformal supergravity, twistors, and the super-BMS group”. In: *Annals of Physics* 171.1 (1986), pp. 52–107. ISSN: 0003-4916. DOI: https://doi.org/10.1016/S0003-4916(86)80023-9. URL: https://www.sciencedirect.com/science/article/pii/S0003491686800239.

[139] Nabamita Banerjee et al. “Extended Supersymmetric BMS_3 algebras and Their Free Field Realisations”. In: *JHEP* 11 (2016), p. 059. DOI: 10.1007/JHEP11(2016)059. arXiv: 1609.09210 [hep-th].

[140] Robert M. Wald and Andreas Zoupas. “A General definition of ’conserved quantities’ in general relativity and other theories of gravity”. In: *Phys. Rev. D* 61 (2000), p. 084027. DOI: 10.1103/PhysRevD.61.084027. arXiv: gr-qc/9911095.

[141] Stephen W. Hawking, Malcolm J. Perry, and Andrew Strominger. “Superrotation Charge and Supertranslation Hair on Black Holes”. In: *JHEP* 05 (2017), p. 161. DOI: 10.1007/JHEP05(2017)161. arXiv: 1611.09175 [hep-th].

[142] Joohan Lee and Robert M. Wald. “Local symmetries and constraints”. In: *Journal of Mathematical Physics* 31.3 (Mar. 1990), pp. 725–743. ISSN: 0022-2488. DOI: 10.1063/1.528801. eprint: https://pubs.aip.org/aip/jmp/article-pdf/31/3/725/19103359/725_1_online.pdf. URL: https://doi.org/10.1063/1.528801.

[143] Adrien Fiorucci. “Leaky covariant phase spaces: Theory and application to Λ-BMS symmetry”. PhD thesis. Brussels U., Intl. Solvay Inst., Brussels, 2021. arXiv: 2112.07666 [hep-th].

[144] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. “The Dynamics of general relativity”. In: *Gen. Rel. Grav.* 40 (2008), pp. 1997–2027. DOI: 10.1007/s10714-008-0661-1. arXiv: gr-qc/0405109.

[145] J. David Brown and Marc Henneaux. "On the Poisson brackets of differentiable generators in classical field theory". In: *Journal of Mathematical Physics* 27.2 (Feb. 1986), pp. 489–491. ISSN: 0022-2488. DOI: 10.1063/1.527249. eprint: https://pubs.aip.org/aip/jmp/article-pdf/27/2/489/19015465/489_1_online.pdf. URL: https://doi.org/10.1063/1.527249.

[146] Patricio Cordero and Claudio Teitelboim. "Hamiltonian treatment of the spherically symmetric einstein-Yang-Mills system". In: *Annals of Physics* 100.1 (1976), pp. 607–631. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(76)90074-9. URL: https://www.sciencedirect.com/science/article/pii/0003491676900749.

[147] R. Benguria, P. Cordero, and C. Teitelboim. "Aspects of the Hamiltonian dynamics of interacting gravitational gauge and Higgs fields with applications to spherical symmetry". In: *Nuclear Physics B* 122.1 (1977), pp. 61–99. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(77)90426-6. URL: https://www.sciencedirect.com/science/article/pii/0550321377904266.

[148] Tullio Regge and Claudio Teitelboim. "Role of surface integrals in the Hamiltonian formulation of general relativity". In: *Annals of Physics* 88.1 (1974), pp. 286–318. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(74)90404-7. URL: https://www.sciencedirect.com/science/article/pii/0003491674904047.

[149] Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems*. Princeton: Princeton University Press, 1992. ISBN: 9780691213866. DOI: doi:10.1515/9780691213866. URL: https://doi.org/10.1515/9780691213866.

[150] Tullio Regge and Claudio Teitelboim. "Role of Surface Integrals in the Hamiltonian Formulation of General Relativity". In: *Annals Phys.* 88 (1974), p. 286. DOI: 10.1016/0003-4916(74)90404-7.

[151] Marc Henneaux and Cédric Troessaert. "Asymptotic symmetries of electromagnetism at spatial infinity". In: *JHEP* 05 (2018), p. 137. DOI: 10.1007/JHEP05(2018)137. arXiv: 1803.10194 [hep-th].

[152] Marc Henneaux and Cédric Troessaert. "BMS Group at Spatial Infinity: the Hamiltonian (ADM) approach". In: *JHEP* 03 (2018), p. 147. DOI: 10.1007/JHEP03(2018)147. arXiv: 1801.03718 [gr-qc].

[153] Marc Henneaux and Cédric Troessaert. "Hamiltonian structure and asymptotic symmetries of the Einstein-Maxwell system at spatial infinity". In: *JHEP* 07 (2018), p. 171. DOI: 10.1007/JHEP07(2018)171. arXiv: 1805.11288 [gr-qc].

[154] Sudarshan Ananth, Lars Brink, and Sucheta Majumdar. "BMS algebra from residual gauge invariance in light-cone gravity". In: *JHEP* 11 (2021), p. 008. DOI: 10.1007/JHEP11(2021)008. arXiv: 2101.00019 [hep-th].

[155] Oscar Fuentealba, Marc Henneaux, and Cédric Troessaert. "Asymptotic Symmetry Algebra of Einstein Gravity and Lorentz Generators". In: *Phys. Rev. Lett.* 131.11 (2023), p. 111402. DOI: 10.1103/PhysRevLett.131.111402. arXiv: 2305.05436 [hep-th].

[156] Oscar Fuentealba et al. "Local supersymmetry and the square roots of Bondi-Metzner-Sachs supertranslations". In: *Phys. Rev. D* 104.12 (2021), p. L121702. DOI: 10.1103/PhysRevD.104.L121702. arXiv: 2108.07825 [hep-th].

[157] Roberto Tanzi and Domenico Giulini. "Asymptotic symmetries of Yang-Mills fields in Hamiltonian formulation". In: *JHEP* 10 (2020), p. 094. DOI: 10.1007/JHEP10(2020)094. arXiv: 2006.07268 [hep-th].

[158] P. A. M. Dirac. "Generalized Hamiltonian Dynamics". In: *Canadian Journal of Mathematics* 2 (1950), pp. 129–148. DOI: 10.4153/CJM-1950-012-1.

[159] M. Henneaux and C. Teitelboim. *Quantization of gauge systems*. 1992. ISBN: 978-0-691-03769-1.

[160] M. M. Sheikh-Jabbari and A. Shirzad. "Boundary conditions as Dirac constraints". In: *Eur. Phys. J. C* 19 (2001), p. 383. DOI: 10.1007/s100520100590. arXiv: hep-th/9907055.

[161] D. M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*. Berlin: Springer-Verlag, 1990. ISBN: 978-3-540-53484-1.

[162] S. W. Hawking. "Particle Creation by Black Holes". In: *Commun. Math. Phys.* 43 (1975). Ed. by G. W. Gibbons and S. W. Hawking. [Erratum: Commun.Math.Phys. 46, 206 (1976)], pp. 199–220. DOI: 10.1007/BF02345020.

[163] J. D. Bekenstein. "Black holes and the second law". In: *Lett. Nuovo Cim.* 4 (1972), pp. 737–740. DOI: 10.1007/BF02757029.

[164] Gerard 't Hooft. "Dimensional reduction in quantum gravity". In: *Conf. Proc. C* 930308 (1993), pp. 284–296. arXiv: gr-qc/9310026.

[165] Leonard Susskind. "The World as a hologram". In: *J. Math. Phys.* 36 (1995), pp. 6377–6396. DOI: 10.1063/1.531249. arXiv: hep-th/9409089.

[166] Juan Martin Maldacena. "The Large N limit of superconformal field theories and supergravity". In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 231–252. DOI: 10.1023/A:1026654312961. arXiv: hep-th/9711200.

[167] Ofer Aharony et al. "Large N field theories, string theory and gravity". In: *Phys. Rept.* 323 (2000), pp. 183–386. DOI: 10.1016/S0370-1573(99)00083-6. arXiv: hep-th/9905111.

[168] S. Perlmutter et al. "Measurements of Ω and Λ from 42 High Redshift Supernovae". In: *Astrophys. J.* 517 (1999), pp. 565–586. DOI: 10.1086/307221. arXiv: astro-ph/9812133.

[169] Adam G. Riess et al. "Observational evidence from supernovae for an accelerating universe and a cosmological constant". In: *Astron. J.* 116 (1998), pp. 1009–1038. DOI: 10.1086/300499. arXiv: astro-ph/9805201.

[170] Leonard Susskind. "Holography in the flat space limit". In: *AIP Conf. Proc.* 493.1 (1999). Ed. by C. P. Burgess and Robert C. Myers, pp. 98–112. DOI: 10.1063/1.1301570. arXiv: hep-th/9901079.

[171] Joseph Polchinski. "S matrices from AdS space-time". In: (Jan. 1999). arXiv: hep-th/9901076.

[172] Steven B. Giddings. "Flat space scattering and bulk locality in the AdS / CFT correspondence". In: *Phys. Rev. D* 61 (2000), p. 106008. DOI: 10.1103/PhysRevD.61.106008. arXiv: hep-th/9907129.

[173] Giovanni Arcioni and Claudio Dappiaggi. "Holography in asymptotically flat space-times and the BMS group". In: *Class. Quant. Grav.* 21 (2004), p. 5655. DOI: 10.1088/0264-9381/21/23/022. arXiv: hep-th/0312186.

[174] Giovanni Arcioni and Claudio Dappiaggi. "Exploring the holographic principle in asymptotically flat space-times via the BMS group". In: *Nucl. Phys. B* 674 (2003), pp. 553–592. DOI: 10.1016/j.nuclphysb.2003.09.051. arXiv: hep-th/0306142.

[175] Robert B. Mann and Donald Marolf. “Holographic renormalization of asymptotically flat spacetimes”. In: *Class. Quant. Grav.* 23 (2006), pp. 2927–2950. DOI: 10.1088/0264-9381/23/9/010. arXiv: hep-th/0511096.

[176] R. K. Sachs. “Gravitational waves in general relativity. 6. The outgoing radiation condition”. In: *Proc. Roy. Soc. Lond. A* 264 (1961), pp. 309–338. DOI: 10.1098/rspa.1961.0202.

[177] Tristan McLoughlin, Andrea Puhm, and Ana-Maria Raclariu. “The SAGEX review on scattering amplitudes chapter 11: soft theorems and celestial amplitudes”. In: *J. Phys. A* 55.44 (2022), p. 443012. DOI: 10.1088/1751-8121/ac9a40. arXiv: 2203.13022 [hep-th].

[178] Daniel Kapec, Monica Pate, and Andrew Strominger. “New Symmetries of QED”. In: *Adv. Theor. Math. Phys.* 21 (2017), pp. 1769–1785. DOI: 10.4310/ATMP.2017.v21.n7.a7. arXiv: 1506.02906 [hep-th].

[179] Sabrina Pasterski, Shu-Heng Shao, and Andrew Strominger. “Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere”. In: *Phys. Rev. D* 96.6 (2017), p. 065026. DOI: 10.1103/PhysRevD.96.065026. arXiv: 1701.00049 [hep-th].

[180] Jan de Boer and Sergey N. Solodukhin. “A Holographic reduction of Minkowski spacetime”. In: *Nucl. Phys. B* 665 (2003), pp. 545–593. DOI: 10.1016/S0550-3213(03)00494-2. arXiv: hep-th/0303006.

[181] Sabrina Pasterski, Shu-Heng Shao, and Andrew Strominger. “Gluon Amplitudes as 2d Conformal Correlators”. In: *Phys. Rev. D* 96.8 (2017), p. 085006. DOI: 10.1103/PhysRevD.96.085006. arXiv: 1706.03917 [hep-th].

[182] Sabrina Pasterski, Monica Pate, and Ana-Maria Raclariu. “Celestial Holography”. In: *2022 Snowmass Summer Study*. Nov. 2021. arXiv: 2111.11392 [hep-th].

[183] Sabrina Pasterski. “Lectures on celestial amplitudes”. In: *Eur. Phys. J. C* 81.12 (2021), p. 1062. DOI: 10.1140/epjc/s10052-021-09846-7. arXiv: 2108.04801 [hep-th].

[184] Laura Donnay, Sabrina Pasterski, and Andrea Puhm. “Asymptotic Symmetries and Celestial CFT”. In: *JHEP* 09 (2020), p. 176. DOI: 10.1007/JHEP09(2020)176. arXiv: 2005.08990 [hep-th].

[185] Wei Fan, Angelos Fotopoulos, and Tomasz R. Taylor. “Soft Limits of Yang-Mills Amplitudes and Conformal Correlators”. In: *JHEP* 05 (2019), p. 121. DOI: 10.1007/JHEP05(2019)121. arXiv: 1903.01676 [hep-th].

[186] Angelos Fotopoulos et al. “Extended BMS Algebra of Celestial CFT”. In: *JHEP* 03 (2020), p. 130. DOI: 10.1007/JHEP03(2020)130. arXiv: 1912.10973 [hep-th].

[187] Shamik Banerjee and Sabrina Pasterski. “Revisiting the Shadow Stress Tensor in Celestial CFT”. In: (Nov. 2022). arXiv: 2212.00257 [hep-th].

[188] Nabamita Banerjee, Tabasum Rahnuma, and Ranveer Kumar Singh. “Asymptotic Symmetry of Four Dimensional Einstein-Yang-Mills and Einstein-Maxwell Theory”. In: (Oct. 2021). arXiv: 2110.15657 [hep-th].

[189] Arjun Bagchi, Prateksh Dhivakar, and Sudipta Dutta. “Holography in Flat Spacetimes: the case for Carroll”. In: (Nov. 2023). arXiv: 2311.11246 [hep-th].

[190] Laura Donnay et al. “Carrollian Perspective on Celestial Holography”. In: *Phys. Rev. Lett.* 129.7 (2022), p. 071602. DOI: 10.1103/PhysRevLett.129.071602. arXiv: 2202.04702 [hep-th].

[191] Laura Donnay et al. "Bridging Carrollian and celestial holography". In: *Phys. Rev. D* 107.12 (2023), p. 126027. DOI: 10.1103/PhysRevD.107.126027. arXiv: 2212.12553 [hep-th].

[192] Kevin Nguyen and Peter West. "Carrollian Conformal Fields and Flat Holography". In: *Universe* 9.9 (2023), p. 385. DOI: 10.3390/universe9090385. arXiv: 2305.02884 [hep-th].

[193] Shamik Banerjee. "Null Infinity and Unitary Representation of The Poincare Group". In: *JHEP* 01 (2019), p. 205. DOI: 10.1007/JHEP01(2019)205. arXiv: 1801.10171 [hep-th].

[194] Arjun Bagchi et al. "Scattering Amplitudes: Celestial and Carrollian". In: *Phys. Rev. Lett.* 128.24 (2022), p. 241601. DOI: 10.1103/PhysRevLett.128.241601. arXiv: 2202.08438 [hep-th].

[195] Arjun Bagchi, Prateksh Dhivakar, and Sudipta Dutta. "AdS Witten diagrams to Carrollian correlators". In: *JHEP* 04 (2023), p. 135. DOI: 10.1007/JHEP04(2023)135. arXiv: 2303.07388 [hep-th].

[196] Arjun Bagchi and Reza Fareghbal. "BMS/GCA Redux: Towards Flatspace Holography from Non-Relativistic Symmetries". In: *JHEP* 10 (2012), p. 092. DOI: 10.1007/JHEP10(2012)092. arXiv: 1203.5795 [hep-th].

[197] Arjun Bagchi et al. "Holography of 3D Flat Cosmological Horizons". In: *Phys. Rev. Lett.* 110.14 (2013), p. 141302. DOI: 10.1103/PhysRevLett.110.141302. arXiv: 1208.4372 [hep-th].

[198] Hamid Afshar et al. "Spin-3 Gravity in Three-Dimensional Flat Space". In: *Phys. Rev. Lett.* 111.12 (2013), p. 121603. DOI: 10.1103/PhysRevLett.111.121603. arXiv: 1307.4768 [hep-th].

[199] Arjun Bagchi, Stephane Detournay, and Daniel Grumiller. "Flat-Space Chiral Gravity". In: *Phys. Rev. Lett.* 109 (2012), p. 151301. DOI: 10.1103/PhysRevLett.109.151301. arXiv: 1208.1658 [hep-th].

[200] Hernan A. Gonzalez et al. "Asymptotically flat spacetimes in three-dimensional higher spin gravity". In: *JHEP* 09 (2013), p. 016. DOI: 10.1007/JHEP09(2013)016. arXiv: 1307.5651 [hep-th].

[201] Arjun Bagchi et al. "Entanglement entropy in Galilean conformal field theories and flat holography". In: *Phys. Rev. Lett.* 114.11 (2015), p. 111602. DOI: 10.1103/PhysRevLett.114.111602. arXiv: 1410.4089 [hep-th].

[202] Jelle Hartong. "Holographic Reconstruction of 3D Flat Space-Time". In: *JHEP* 10 (2016), p. 104. DOI: 10.1007/JHEP10(2016)104. arXiv: 1511.01387 [hep-th].

[203] Arjun Bagchi, Daniel Grumiller, and Wout Merbis. "Stress tensor correlators in three-dimensional gravity". In: *Phys. Rev. D* 93.6 (2016), p. 061502. DOI: 10.1103/PhysRevD.93.061502. arXiv: 1507.05620 [hep-th].

[204] Hongliang Jiang, Wei Song, and Qiang Wen. "Entanglement Entropy in Flat Holography". In: *JHEP* 07 (2017), p. 142. DOI: 10.1007/JHEP07(2017)142. arXiv: 1706.07552 [hep-th].

[205] Eliot Hijano and Charles Rabideau. "Holographic entanglement and Poincaré blocks in three-dimensional flat space". In: *JHEP* 05 (2018), p. 068. DOI: 10.1007/JHEP05(2018)068. arXiv: 1712.07131 [hep-th].

[206] Arjun Bagchi et al. "Flat Holography: Aspects of the dual field theory". In: *JHEP* 12 (2016), p. 147. DOI: 10.1007/JHEP12(2016)147. arXiv: 1609.06203 [hep-th].

[207] Zvi Bern. "Perturbative quantum gravity and its relation to gauge theory". In: *Living Rev. Rel.* 5 (2002), p. 5. DOI: 10.12942/lrr-2002-5. arXiv: gr-qc/0206071.

[208] H. Kawai, D. C. Lewellen, and S. H. H. Tye. "A Relation Between Tree Amplitudes of Closed and Open Strings". In: *Nucl. Phys. B* 269 (1986), pp. 1–23. DOI: 10.1016/0550-3213(86)90362-7.

[209] Z. Bern, J. J. M. Carrasco, and Henrik Johansson. "New Relations for Gauge-Theory Amplitudes". In: *Phys. Rev. D* 78 (2008), p. 085011. DOI: 10.1103/PhysRevD.78.085011. arXiv: 0805.3993 [hep-ph].

[210] Freddy Cachazo, Song He, and Ellis Ye Yuan. "Scattering equations and Kawai-Lewellen-Tye orthogonality". In: *Phys. Rev. D* 90.6 (2014), p. 065001. DOI: 10.1103/PhysRevD.90.065001. arXiv: 1306.6575 [hep-th].

[211] Freddy Cachazo, Song He, and Ellis Ye Yuan. "Scattering of Massless Particles in Arbitrary Dimensions". In: *Phys. Rev. Lett.* 113.17 (2014), p. 171601. DOI: 10.1103/PhysRevLett.113.171601. arXiv: 1307.2199 [hep-th].

[212] Tim Adamo et al. "Snowmass White Paper: the Double Copy and its Applications". In: *2022 Snowmass Summer Study*. Apr. 2022. arXiv: 2204.06547 [hep-th].

[213] Zvi Bern, John Joseph M. Carrasco, and Henrik Johansson. "Perturbative Quantum Gravity as a Double Copy of Gauge Theory". In: *Phys. Rev. Lett.* 105 (6 Aug. 2010), p. 061602. DOI: 10.1103/PhysRevLett.105.061602. URL: https://link.aps.org/doi/10.1103/PhysRevLett.105.061602.

[214] Zheng-Wen Liu. "Soft theorems in maximally supersymmetric theories". In: *Eur. Phys. J. C* 75.3 (2015), p. 105. DOI: 10.1140/epjc/s10052-015-3304-1. arXiv: 1410.1616 [hep-th].

[215] Massimo Bianchi, Henriette Elvang, and Daniel Z. Freedman. "Generating Tree Amplitudes in N=4 SYM and N = 8 SG". In: *JHEP* 09 (2008), p. 063. DOI: 10.1088/1126-6708/2008/09/063. arXiv: 0805.0757 [hep-th].

[216] Z. Bern et al. "Multileg one loop gravity amplitudes from gauge theory". In: *Nucl. Phys. B* 546 (1999), pp. 423–479. DOI: 10.1016/S0550-3213(99)00029-2. arXiv: hep-th/9811140.

[217] Z. Bern et al. "One loop n point helicity amplitudes in (selfdual) gravity". In: *Phys. Lett. B* 444 (1998), pp. 273–283. DOI: 10.1016/S0370-2693(98)01397-5. arXiv: hep-th/9809160.

[218] John Golden and Marcus Spradlin. "Collinear and Soft Limits of Multi-Loop Integrands in N=4 Yang-Mills". In: *JHEP* 05 (2012), p. 027. DOI: 10.1007/JHEP05(2012)027. arXiv: 1203.1915 [hep-th].

[219] Jacob L. Bourjaily et al. "The Soft-Collinear Bootstrap: N=4 Yang-Mills Amplitudes at Six and Seven Loops". In: *JHEP* 03 (2012), p. 032. DOI: 10.1007/JHEP03(2012)032. arXiv: 1112.6432 [hep-th].

[220] Dhritiman Nandan and Congkao Wen. "Generating All Tree Amplitudes in N=4 SYM by Inverse Soft Limit". In: *JHEP* 08 (2012), p. 040. DOI: 10.1007/JHEP08(2012)040. arXiv: 1204.4841 [hep-th].

[221] Hongliang Jiang. "Celestial superamplitude in $\mathcal{N} = 4$ SYM theory". In: *JHEP* 08 (2021), p. 031. DOI: 10.1007/JHEP08(2021)031. arXiv: 2105.10269 [hep-th].

[222] Mirah Gary and Steven B. Giddings. "The Flat space S-matrix from the AdS/CFT correspondence?" In: *Phys. Rev. D* 80 (2009), p. 046008. DOI: 10.1103/PhysRevD.80.046008. arXiv: 0904.3544 [hep-th].

[223] Shota Komatsu et al. "Landau diagrams in AdS and S-matrices from conformal correlators". In: *JHEP* 11 (2020), p. 046. DOI: 10.1007/JHEP11(2020)046. arXiv: 2007.13745 [hep-th].

[224] Eliot Hijano. "Flat space physics from AdS/CFT". In: *JHEP* 07 (2019), p. 132. DOI: 10.1007/JHEP07(2019)132. arXiv: 1905.02729 [hep-th].

[225] Joao Penedones. "Writing CFT correlation functions as AdS scattering amplitudes". In: *JHEP* 03 (2011), p. 025. DOI: 10.1007/JHEP03(2011)025. arXiv: 1011.1485 [hep-th].

[226] A. Liam Fitzpatrick et al. "A Natural Language for AdS/CFT Correlators". In: *JHEP* 11 (2011), p. 095. DOI: 10.1007/JHEP11(2011)095. arXiv: 1107.1499 [hep-th].

[227] Leonardo Rastelli and Xinan Zhou. "Mellin amplitudes for $AdS_5 \times S^5$". In: *Phys. Rev. Lett.* 118.9 (2017), p. 091602. DOI: 10.1103/PhysRevLett.118.091602. arXiv: 1608.06624 [hep-th].

[228] Edward Witten. "Anti-de Sitter space and holography". In: *Adv. Theor. Math. Phys.* 2 (1998), pp. 253–291. DOI: 10.4310/ATMP.1998.v2.n2.a2. arXiv: hep-th/9802150.

[229] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. "Gauge theory correlators from noncritical string theory". In: *Phys. Lett. B* 428 (1998), pp. 105–114. DOI: 10.1016/S0370-2693(98)00377-3. arXiv: hep-th/9802109.

[230] A. Liam Fitzpatrick and Jared Kaplan. "Analyticity and the Holographic S-Matrix". In: *JHEP* 10 (2012), p. 127. DOI: 10.1007/JHEP10(2012)127. arXiv: 1111.6972 [hep-th].

[231] Miguel F. Paulos et al. "The S-matrix bootstrap. Part I: QFT in AdS". In: *JHEP* 11 (2017), p. 133. DOI: 10.1007/JHEP11(2017)133. arXiv: 1607.06109 [hep-th].

[232] Sarthak Duary, Eliot Hijano, and Milan Patra. "Towards an IR finite S-matrix in the flat limit of AdS/CFT". In: (Nov. 2022). arXiv: 2211.13711 [hep-th].

[233] A. Liam Fitzpatrick and Jared Kaplan. "Scattering States in AdS/CFT". In: (Apr. 2011). arXiv: 1104.2597 [hep-th].

[234] Yue-Zhou Li. "Notes on flat-space limit of AdS/CFT". In: *JHEP* 09 (2021), p. 027. DOI: 10.1007/JHEP09(2021)027. arXiv: 2106.04606 [hep-th].

[235] Alex Hamilton et al. "Holographic representation of local bulk operators". In: *Phys. Rev. D* 74 (2006), p. 066009. DOI: 10.1103/PhysRevD.74.066009. arXiv: hep-th/0606141.

[236] Abhijit Gadde and Trakshu Sharma. "A scattering amplitude for massive particles in AdS". In: *JHEP* 09 (2022), p. 157. DOI: 10.1007/JHEP09(2022)157. arXiv: 2204.06462 [hep-th].

[237] Nabamita Banerjee, Tabasum Rahnuma, and Ranveer Kumar Singh. "Soft and collinear limits in $\mathcal{N} = 8$ supergravity using double copy formalism". In: *JHEP* 04 (2023), p. 126. DOI: 10.1007/JHEP04(2023)126. arXiv: 2212.11480 [hep-th].

[238] Nabamita Banerjee, Tabasum Rahnuma, and Ranveer Kumar Singh. "Asymptotic Symmetry algebra of $\mathcal{N} = 8$ Supergravity". In: (Dec. 2022). arXiv: 2212.12133 [hep-th].

[239] Nabamita Banerjee, Arindam Bhattacharjee, and Arpita Mitra. "Classical Soft Theorem in the AdS-Schwarzschild spacetime in small cosmological constant limit". In: *JHEP* 01 (2021), p. 038. DOI: 10.1007/JHEP01(2021)038. arXiv: 2008.02828 [hep-th].

[240] Nabamita Banerjee, Karan Fernandes, and Arpita Mitra. "Soft photon theorem in the small negative cosmological constant limit". In: *JHEP* 08 (2021), p. 105. DOI: 10.1007/JHEP08(2021)105. arXiv: 2102.06165 [hep-th].

[241] Nabamita Banerjee, Karan Fernandes, and Arpita Mitra. "$1/L^2$ corrected soft photon theorem from a CFT_3 Ward identity". In: *JHEP* 04 (2023), p. 055. DOI: 10.1007/JHEP04(2023)055. arXiv: 2209.06802 [hep-th].

[242] Karan Fernandes, Nabamita Banerjee, and Arpita Mitra. "Soft factors with AdS radius corrections". In: *JHAP* 3.4 (2023), pp. 5–22. DOI: 10.22128/jhap.2023.738.1059. arXiv: 2310.19299 [hep-th].

[243] Luc Blanchet et al. "Multipole expansion of gravitational waves: memory effects and Bondi aspects". In: *JHEP* 07 (2023), p. 123. DOI: 10.1007/JHEP07(2023)123. arXiv: 2303.07732 [gr-qc].

[244] Lydia Bieri and David Garfinkle. "Perturbative and gauge invariant treatment of gravitational wave memory". In: *Phys. Rev. D* 89 (8 Apr. 2014), p. 084039. DOI: 10.1103/PhysRevD.89.084039. URL: https://link.aps.org/doi/10.1103/PhysRevD.89.084039.

[245] Vladimir B. Braginsky and Kip S. Thorne. "Gravitational-wave bursts with memory and experimental prospects". In: *Nature* 327.6118 (May 1987), pp. 123–125. ISSN: 1476-4687. DOI: 10.1038/327123a0. URL: https://doi.org/10.1038/327123a0.

[246] Marc Favata. "Post-Newtonian corrections to the gravitational-wave memory for quasi-circular, inspiralling compact binaries". In: *Phys. Rev. D* 80.2 (2009), p. 024002. DOI: 10.1103/PhysRevD.80.024002. arXiv: 0812.0069 [gr-qc].

[247] David A. Nichols. "Center-of-mass angular momentum and memory effect in asymptotically flat spacetimes". In: *Phys. Rev. D* 98.6 (2018), p. 064032. DOI: 10.1103/PhysRevD.98.064032. arXiv: 1807.08767 [gr-qc].

[248] David A. Nichols. "Spin memory effect for compact binaries in the post-Newtonian approximation". In: *Phys. Rev. D* 98.10 (2018), p. 104023. DOI: 10.1103/PhysRevD.98.104023. arXiv: 1808.01288 [gr-qc].

[249] Éanna É. Flanagan et al. "A new measure of the center-of-mass of isolated systems in general relativity". In: *Phys. Rev. D* 99.8 (2019), p. 084044. DOI: 10.1103/PhysRevD.99.084044. arXiv: 1902.09424 [gr-qc].

[250] Luc Blanchet et al. "Gravitational radiation from inspiralling compact binaries completed at the third post-Newtonian order". In: *Phys. Rev. Lett.* 99.20 (2009), p. 201102. DOI: 10.1103/PhysRevLett.99.201102. arXiv: 0907.2694 [gr-qc].

[251] Luc Blanchet and Thibault Damour. "Tail Transported Temporal Correlations in the Dynamics of a Gravitating System". In: *Phys. Rev. D* 37 (1988), pp. 1410–1435. DOI: 10.1103/PhysRevD.37.1410.

[252] Felix A. E. Pirani. "Invariant Formulation of Gravitational Radiation Theory". In: *Phys. Rev.* 105 (1957), pp. 1089–1099. DOI: 10.1103/PhysRev.105.1089.

[253] Vladimir B. Braginskii and Leonid P. Grishchuk. “Kinematic resonance and the memory effect in free mass gravitational antennas”. In: *Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki* 89 (1985), pp. 744–750.

[254] Geoffrey Compère, Roberto Oliveri, and Ali Seraj. “The Poincaré and BMS flux-balance laws with application to binary systems”. In: *JHEP* 10 (2020). [Erratum: JHEP 06, 045 (2024)], p. 116. DOI: 10.1007/JHEP10(2020)116. arXiv: 1912.03164 [gr-qc].

[255] Luc Blanchet. “Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries”. In: *Living Rev. Rel.* 17 (2014), p. 2. DOI: 10.12942/lrr-2014-2. arXiv: 1310.1528 [gr-qc].

[256] C. M. Will and A. G. Wiseman. “Gravitational radiation from compact binary systems: Gravitational wave forms and energy loss to second post-Newtonian order”. In: *Phys. Rev. D* 54 (1996), pp. 4813–4848. arXiv: gr-qc/9608012.

[257] Michèle Levi. “Effective Field Theories of Post-Newtonian Gravity: A comprehensive review”. In: *Rept. Prog. Phys.* 83.7 (2020), p. 075901. DOI: 10.1088/1361-6633/ab12bc. arXiv: 1807.01699 [hep-th].

[258] Rafael A. Porto. “The effective field theorist’s approach to gravitational dynamics”. In: *Phys. Rept.* 633 (2016), pp. 1–104. DOI: 10.1016/j.physrep.2016.04.003. arXiv: 1601.04914 [hep-th].

[259] Stefano Foffa et al. “Static two-body potential at fifth post-Newtonian order”. In: *Phys. Rev. Lett.* 122.24 (2019), p. 241605. DOI: 10.1103/PhysRevLett.122.241605. arXiv: 1902.10571 [gr-qc].

[260] S. Laporta. “High precision calculation of multiloop Feynman integrals by difference equations”. In: *Int. J. Mod. Phys. A* 15 (2000), pp. 5087–5159. DOI: 10.1142/S0217751X00002159. arXiv: hep-ph/0102033.

[261] K. G. Chetyrkin and F. V. Tkachov. “Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops”. In: *Nucl. Phys. B* 192 (1981), pp. 159–204. DOI: 10.1016/0550-3213(81)90199-1.

[262] K. S. Thorne. “Multipole Expansions of Gravitational Radiation”. In: *Rev. Mod. Phys.* 52 (1980), pp. 299–339.

[263] R. V. Wagoner and C. M. Will. “Post-Newtonian gravitational radiation from orbiting point masses”. In: *Astrophys. J.* 210 (1976), pp. 764–775.

[264] L. Blanchet and T. Damour. “Radiative gravitational field in general relativity I. General structure of the field outside the source”. In: *Phil. Trans. Roy. Soc. Lond. A* 320 (1986), pp. 379–430.

[265] Walter D. Goldberger and Ira Z. Rothstein. “An Effective field theory of gravity for extended objects”. In: *Phys. Rev. D* 73 (2006), p. 104029. DOI: 10.1103/PhysRevD.73.104029. arXiv: hep-th/0409156.

[266] Zvi Bern et al. “Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at O(G4)”. In: *Phys. Rev. Lett.* 128.16 (2022), p. 161103. DOI: 10.1103/PhysRevLett.128.161103. arXiv: 2112.10750 [hep-th].

[267] Zvi Bern et al. “Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$”. In: *Phys. Rev. Lett.* 126.17 (2021), p. 171601. DOI: 10.1103/PhysRevLett.126.171601. arXiv: 2101.07254 [hep-th].

[268] Clifford Cheung, Ira Z. Rothstein, and Mikhail P. Solon. "From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion". In: *Phys. Rev. Lett.* 121.25 (2018), p. 251101. DOI: 10.1103/PhysRevLett.121.251101. arXiv: 1808.02489 [hep-th].

[269] Zhengwen Liu, Rafael A. Porto, and Zixin Yang. "Spin Effects in the Effective Field Theory Approach to Post-Minkowskian Conservative Dynamics". In: *JHEP* 06 (2021), p. 012. DOI: 10.1007/JHEP06(2021)012. arXiv: 2102.10059 [hep-th].

[270] Gregor Kälin and Rafael A. Porto. "Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics". In: *JHEP* 11 (2020), p. 106. DOI: 10.1007/JHEP11(2020)106. arXiv: 2006.01184 [hep-th].

[271] Gregor Kälin, Zhengwen Liu, and Rafael A. Porto. "Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach". In: *Phys. Rev. Lett.* 125.26 (2020), p. 261103. DOI: 10.1103/PhysRevLett.125.261103. arXiv: 2007.04977 [hep-th].

[272] Christoph Dlapa et al. "Dynamics of binary systems to fourth Post-Minkowskian order from the effective field theory approach". In: *Phys. Lett. B* 831 (2022), p. 137203. DOI: 10.1016/j.physletb.2022.137203. arXiv: 2106.08276 [hep-th].

[273] Geoffrey Compère. *Advanced Lectures on General Relativity*. Vol. 952. Cham, Switzerland: Springer, Cham, Feb. 2019. ISBN: 978-3-030-04259-2, 978-3-030-04260-8. DOI: 10.1007/978-3-030-04260-8.

[274] Geoffrey Compère, R. Oliveri, and A. Seraj. "Gravitational multipole moments from Noether charges". In: *JHEP* 05 (2018), p. 054. DOI: 10.1007/JHEP05(2018)054. arXiv: 1711.08806 [hep-th].

[275] Geoffrey Compère, Samuel E. Gralla, and Hongji Wei. "An asymptotic framework for gravitational scattering". In: *Class. Quant. Grav.* 40.20 (2023), p. 205018. DOI: 10.1088/1361-6382/acf5c1. arXiv: 2303.17124 [gr-qc].

[276] Geoffrey Compère, Roberto Oliveri, and Ali Seraj. "Metric reconstruction from celestial multipoles". In: *JHEP* 11 (2022), p. 001. DOI: 10.1007/JHEP11(2022)001. arXiv: 2206.12597 [hep-th].

[277] David A. Nichols. "Center-of-mass angular momentum and memory effect in asymptotically flat spacetimes". In: *Phys. Rev. D* 98.6 (2018), p. 064032. DOI: 10.1103/PhysRevD.98.064032. arXiv: 1807.08767 [gr-qc].

[278] David A. Nichols. "Spin memory effect for compact binaries in the post-Newtonian approximation". In: *Phys. Rev. D* 95.8 (2017), p. 084048. DOI: 10.1103/PhysRevD.95.084048. arXiv: 1702.03300 [gr-qc].

[279] Sabrina Pasterski, Andrew Strominger, and Alexander Zhiboedov. "New Gravitational Memories". In: *JHEP* 12 (2016), p. 053. DOI: 10.1007/JHEP12(2016)053. arXiv: 1502.06120 [hep-th].

[280] Reinhard A. Breuer and Ekkart Rudolph. "The force law for the dynamic two-body problem in the second post-Newtonian approximation of general relativity". In: *General Relativity and Gravitation* 14.2 (Feb. 1982), pp. 181–211. ISSN: 1572-9532. DOI: 10.1007/BF00756922. URL: https://doi.org/10.1007/BF00756922.

[281] Thibaut Damour and Nathalie Deruelle. "Generalized lagrangian of two point masses in the post-post newtonian approximation of general relativity". In: *CR Seances Acad. Sci., Ser. 2;(France)* 293.8 (1981).

[282] Thibaut Damour and Nathalie Deruelle. "Radiation reaction and angular momentum loss in small angle gravitational scattering". In: *Physics Letters A* 87.3 (1981), pp. 81–84. ISSN: 0375-9601. DOI: https://doi.org/10.1016/0375-9601(81)90567-3. URL: https://www.sciencedirect.com/science/article/pii/0375960181905673.

[283] Éanna É. Flanagan et al. "Persistent gravitational wave observables: general framework". In: *Phys. Rev. D* 99.8 (2019), p. 084044. DOI: 10.1103/PhysRevD.99.084044. arXiv: 1901.00021 [gr-qc].

[284] B. P. Abbott et al. "Observation of Gravitational Waves from a Binary Black Hole Merger". In: *Phys. Rev. Lett.* 116.6 (2016), p. 061102. DOI: 10.1103/PhysRevLett.116.061102. arXiv: 1602.03837 [gr-qc].

[285] R. Abbott et al. "GWTC-3: Compact Binary Coalescences Observed by LIGO and Virgo During the Second Part of the Third Observing Run". In: (Nov. 2021). arXiv: 2111.03606 [gr-qc].

[286] J. Aasi et al. "Advanced LIGO". In: *Class. Quant. Grav.* 32 (2015), p. 074001. DOI: 10.1088/0264-9381/32/7/074001. arXiv: 1411.4547 [gr-qc].

[287] F. Acernese et al. "Advanced Virgo: a second-generation interferometric gravitational wave detector". In: *Class. Quant. Grav.* 32.2 (2015), p. 024001. DOI: 10.1088/0264-9381/32/2/024001. arXiv: 1408.3978 [gr-qc].

[288] T. Akutsu et al. "Overview of KAGRA: Calibration, detector characterization, physical environmental monitors, and the geophysics interferometer". In: *PTEP* 2021.5 (2021), 05A102. DOI: 10.1093/ptep/ptab018. arXiv: 2009.09305 [gr-qc].

[289] Benjamin P Abbott et al. "Exploring the Sensitivity of Next Generation Gravitational Wave Detectors". In: *Class. Quant. Grav.* 34.4 (2017), p. 044001. DOI: 10.1088/1361-6382/aa51f4. arXiv: 1607.08697 [astro-ph.IM].

[290] David Reitze et al. "Cosmic Explorer: The U.S. Contribution to Gravitational-Wave Astronomy beyond LIGO". In: *Bull. Am. Astron. Soc.* 51.7 (2019), p. 035. arXiv: 1907.04833 [astro-ph.IM].

[291] M. Punturo et al. "The Einstein Telescope: A third-generation gravitational wave observatory". In: *Class. Quant. Grav.* 27 (2010). Ed. by Fulvio Ricci, p. 194002. DOI: 10.1088/0264-9381/27/19/194002.

[292] "LISA collaboration, Laser Interferometer Space Antenna". In: (). eprint: 1702.00786.

[293] Poul H. Damgaard et al. "Classical observables from the exponential representation of the gravitational S-matrix". In: *JHEP* 09 (2023), p. 183. DOI: 10.1007/JHEP09(2023)183. arXiv: 2307.04746 [hep-th].

[294] Gustav Mogull, Jan Plefka, and Jan Steinhoff. "Classical black hole scattering from a worldline quantum field theory". In: *JHEP* 02 (2021), p. 048. DOI: 10.1007/JHEP02(2021)048. arXiv: 2010.02865 [hep-th].

[295] Gustav Uhre Jakobsen et al. "All things retarded: radiation-reaction in worldline quantum field theory". In: *JHEP* 10 (2022), p. 128. DOI: 10.1007/JHEP10(2022)128. arXiv: 2207.00569 [hep-th].

[296] Gustav Uhre Jakobsen and Gustav Mogull. “Conservative and Radiative Dynamics of Spinning Bodies at Third Post-Minkowskian Order Using Worldline Quantum Field Theory”. In: *Phys. Rev. Lett.* 128.14 (2022), p. 141102. DOI: 10.1103/PhysRevLett.128.141102. arXiv: 2201.07778 [hep-th].

[297] Gustav Uhre Jakobsen et al. “Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory”. In: *Phys. Rev. Lett.* 126.20 (2021), p. 201103. DOI: 10.1103/PhysRevLett.126.201103. arXiv: 2101.12688 [gr-qc].

[298] Thomas W. Baumgarte and Stuart L. Shapiro. *Numerical Relativity: Solving Einstein’s Equations on the Computer*. Cambridge University Press, 2010.

[299] Miguel Alcubierre. *Introduction to 3+1 Numerical Relativity*. Oxford University Press, Apr. 2008. ISBN: 9780199205677. DOI: 10.1093/acprof:oso/9780199205677.001.0001. URL: https://doi.org/10.1093/acprof:oso/9780199205677.001.0001.

[300] Frans Pretorius. “Evolution of binary black hole spacetimes”. In: *Phys. Rev. Lett.* 95 (2005), p. 121101. DOI: 10.1103/PhysRevLett.95.121101. arXiv: gr-qc/0507014.

[301] Manuela Campanelli et al. “Accurate evolutions of orbiting black-hole binaries without excision”. In: *Phys. Rev. Lett.* 96 (2006), p. 111101. DOI: 10.1103/PhysRevLett.96.111101. arXiv: gr-qc/0511048.

[302] John G. Baker et al. “Gravitational wave extraction from an inspiraling configuration of merging black holes”. In: *Phys. Rev. Lett.* 96 (2006), p. 111102. DOI: 10.1103/PhysRevLett.96.111102. arXiv: gr-qc/0511103.

[303] Piero Rettegno et al. “Strong-field scattering of two spinning black holes: Numerical Relativity versus post-Minkowskian gravity”. In: (July 2023). arXiv: 2307.06999 [gr-qc].

[304] Thibault Damour and Piero Rettegno. “Strong-field scattering of two black holes: Numerical relativity meets post-Minkowskian gravity”. In: *Phys. Rev. D* 107.6 (2023), p. 064051. DOI: 10.1103/PhysRevD.107.064051. arXiv: 2211.01399 [gr-qc].

[305] Francois Foucart et al. “Snowmass2021 Cosmic Frontier White Paper: Numerical relativity for next-generation gravitational-wave probes of fundamental physics”. In: (Mar. 2022). arXiv: 2203.08139 [gr-qc].

[306] Xiaolin Liu, Zhoujian Cao, and Zong-Hong Zhu. “A higher-multipole gravitational waveform model for an eccentric binary black holes based on the effective-one-body-numerical-relativity formalism”. In: (Feb. 2021). arXiv: 2102.08614 [gr-qc].

[307] Steven Giddings and Sean Weinberg. “Gauge-invariant observables in gravity and electromagnetism: black hole backgrounds and null dressings”. In: *Phys. Rev. D* 102.2 (2020), p. 026010. DOI: 10.1103/PhysRevD.102.026010. arXiv: 1911.09115 [hep-th].

[308] Kouji Nakamura. “Proposal of a gauge-invariant treatment of l = 0, 1-mode perturbations on Schwarzschild background spacetime”. In: *Class. Quant. Grav.* 38.14 (2021), p. 145010. DOI: 10.1088/1361-6382/ac03f9. arXiv: 2102.00830 [gr-qc].

[309] Kouji Nakamura. “Construction of gauge-invariant variables of linear metric perturbations on an arbitrary background spacetime”. In: *PTEP* 2013 (2013), 043E02. DOI: 10.1093/ptep/ptt006. arXiv: 1105.4007 [gr-qc].

[310] Kouji Nakamura. “Recursive structure in the definitions of gauge-invariant variables for any order perturbations”. In: *Class. Quant. Grav.* 31 (2014), p. 135013. DOI: 10.1088/0264-9381/31/13/135013. arXiv: 1403.1004 [gr-qc].

[311] Kouji Nakamura. “General formulation of general-relativistic higher-order gauge-invariant perturbation theory”. In: *Class. Quant. Grav.* 28 (2011), p. 122001. DOI: 10.1088/0264-9381/28/12/122001. arXiv: 1011.5272 [gr-qc].

[312] Nigel T. Bishop and Luciano Rezzolla. “Extraction of Gravitational Waves in Numerical Relativity”. In: *Living Rev. Rel.* 19 (2016), p. 2. DOI: 10.1007/s41114-016-0001-9. arXiv: 1606.02532 [gr-qc].

[313] A. M. Abrahams et al. “Gravitational wave extraction and outer boundary conditions by perturbative matching”. In: *Phys. Rev. Lett.* 80 (1998), pp. 1812–1815. DOI: 10.1103/PhysRevLett.80.1812. arXiv: gr-qc/9709082.

[314] Luciano Rezzolla et al. “Cauchy perturbative matching and outer boundary conditions: Computational studies”. In: *Phys. Rev. D* 59 (1999), p. 064001. DOI: 10.1103/PhysRevD.59.064001. arXiv: gr-qc/9807047.

[315] S. W. Hawking. “Breakdown of predictability in gravitational collapse”. In: *Phys. Rev. D* 14 (10 Nov. 1976), pp. 2460–2473. DOI: 10.1103/PhysRevD.14.2460. URL: https://link.aps.org/doi/10.1103/PhysRevD.14.2460.

[316] Ahmed Almheiri et al. “Black Holes: Complementarity or Firewalls?” In: *JHEP* 02 (2013), p. 062. DOI: 10.1007/JHEP02(2013)062. arXiv: 1207.3123 [hep-th].

[317] Yasunori Nomura, Jaime Varela, and Sean J. Weinberg. “Black holes or firewalls: A theory of horizons”. In: *Phys. Rev. D* 88 (8 Oct. 2013), p. 084052. DOI: 10.1103/PhysRevD.88.084052. URL: https://link.aps.org/doi/10.1103/PhysRevD.88.084052.

[318] Geoffrey Penington. “Entanglement Wedge Reconstruction and the Information Paradox”. In: *JHEP* 09 (2020), p. 002. DOI: 10.1007/JHEP09(2020)002. arXiv: 1905.08255 [hep-th].

[319] Veronika E. Hubeny, Mukund Rangamani, and Tadashi Takayanagi. “A Covariant holographic entanglement entropy proposal”. In: *JHEP* 07 (2007), p. 062. DOI: 10.1088/1126-6708/2007/07/062. arXiv: 0705.0016 [hep-th].

[320] Shinsei Ryu and Tadashi Takayanagi. “Holographic derivation of entanglement entropy from AdS/CFT”. In: *Phys. Rev. Lett.* 96 (2006), p. 181602. DOI: 10.1103/PhysRevLett.96.181602. arXiv: hep-th/0603001.

[321] Shinsei Ryu and Tadashi Takayanagi. “Aspects of Holographic Entanglement Entropy”. In: *JHEP* 08 (2006), p. 045. DOI: 10.1088/1126-6708/2006/08/045. arXiv: hep-th/0605073.

[322] Adam R. Brown et al. “Holographic Complexity Equals Bulk Action?” In: *Phys. Rev. Lett.* 116.19 (2016), p. 191301. DOI: 10.1103/PhysRevLett.116.191301. arXiv: 1509.07876 [hep-th].

[323] Adam R. Brown et al. “Holographic Complexity Equals Bulk Action?” In: *Phys. Rev. Lett.* 116 (19 May 2016), p. 191301. DOI: 10.1103/PhysRevLett.116.191301. URL: https://link.aps.org/doi/10.1103/PhysRevLett.116.191301.

[324] Douglas Stanford and Leonard Susskind. “Complexity and shock wave geometries”. In: *Phys. Rev. D* 90 (12 Dec. 2014), p. 126007. DOI: 10.1103/PhysRevD.90.126007. URL: https://link.aps.org/doi/10.1103/PhysRevD.90.126007.

[325] Leonard Susskind. “Computational Complexity and Black Hole Horizons”. In: *Fortsch. Phys.* 64 (2016). [Addendum: Fortsch.Phys. 64, 44–48 (2016)], pp. 24–43. DOI: 10.1002/prop.201500092. arXiv: 1403.5695 [hep-th].

[326] Andrew Strominger and Cumrun Vafa. "Microscopic origin of the Bekenstein-Hawking entropy". In: *Phys. Lett. B* 379 (1996), pp. 99–104. DOI: 10.1016/0370-2693(96)00345-0. arXiv: hep-th/9601029.

[327] Juan Martin Maldacena, Andrew Strominger, and Edward Witten. "Black hole entropy in M theory". In: *JHEP* 12 (1997), p. 002. DOI: 10.1088/1126-6708/1997/12/002. arXiv: hep-th/9711053.

[328] Ashoke Sen. "Entropy Function and AdS(2) / CFT(1) Correspondence". In: *JHEP* 11 (2008), p. 075. DOI: 10.1088/1126-6708/2008/11/075. arXiv: 0805.0095 [hep-th].

[329] J. David Brown and M. Henneaux. "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity". In: *Commun. Math. Phys.* 104 (1986), pp. 207–226. DOI: 10.1007/BF01211590.

[330] Monica Guica et al. "The Kerr/CFT Correspondence". In: *Phys. Rev. D* 80 (2009), p. 124008. DOI: 10.1103/PhysRevD.80.124008. arXiv: 0809.4266 [hep-th].

[331] Ali Seraj. "Conserved charges, surface degrees of freedom, and black hole entropy". PhD thesis. IPM, Tehran, 2016. arXiv: 1603.02442 [hep-th].

[332] Steven Carlip. "Entropy from conformal field theory at Killing horizons". In: *Class. Quant. Grav.* 16 (1999), pp. 3327–3348. DOI: 10.1088/0264-9381/16/10/322. arXiv: gr-qc/9906126.

[333] Stephen W. Hawking, Malcolm J. Perry, and Andrew Strominger. "Soft Hair on Black Holes". In: *Phys. Rev. Lett.* 116.23 (2016), p. 231301. DOI: 10.1103/PhysRevLett.116.231301. arXiv: 1601.00921 [hep-th].

[334] Adel A. Rahman and Robert M. Wald. "Black Hole Memory". In: *Phys. Rev. D* 101.12 (2020), p. 124010. DOI: 10.1103/PhysRevD.101.124010. arXiv: 1912.12806 [gr-qc].

[335] Laura Donnay et al. "Black hole memory effect". In: *Phys. Rev. D* 98.12 (2018), p. 124016. DOI: 10.1103/PhysRevD.98.124016. arXiv: 1809.07266 [hep-th].

[336] Venkatesa Chandrasekaran, Éanna É. Flanagan, and Kartik Prabhu. "Symmetries and charges of general relativity at null boundaries". In: *JHEP* 11 (2018). [Erratum: JHEP 07, 224 (2023)], p. 125. DOI: 10.1007/JHEP11(2018)125. arXiv: 1807.11499 [hep-th].

[337] Glenn Barnich et al. "Asymptotic symmetries and dynamics of three-dimensional flat supergravity". In: *JHEP* 08 (2014), p. 071. DOI: 10.1007/JHEP08(2014)071. arXiv: 1407.4275 [hep-th].

[338] Ivano Lodato and Wout Merbis. "Super-BMS$_3$ algebras from $\mathcal{N} = 2$ flat supergravities". In: *JHEP* 11 (2016), p. 150. DOI: 10.1007/JHEP11(2016)150. arXiv: 1610.07506 [hep-th].

[339] Nabamita Banerjee, Ivano Lodato, and Turmoli Neogi. "N=4 Supersymmetric BMS3 algebras from asymptotic symmetry analysis". In: *Phys. Rev. D* 96.6 (2017), p. 066029. DOI: 10.1103/PhysRevD.96.066029. arXiv: 1706.02922 [hep-th].

[340] Oscar Fuentealba, Javier Matulich, and Ricardo Troncoso. "Asymptotic structure of $\mathcal{N} = 2$ supergravity in 3D: extended super-BMS$_3$ and nonlinear energy bounds". In: *JHEP* 09 (2017), p. 030. DOI: 10.1007/JHEP09(2017)030. arXiv: 1706.07542 [hep-th].

[341] Nabamita Banerjee et al. "Maximally $\mathcal{N}$ -extended super-BMS$_3$ algebras and generalized 3D gravity solutions". In: *JHEP* 01 (2019), p. 115. DOI: 10.1007/JHEP01(2019)115. arXiv: 1807.06768 [hep-th].

[342] Nabamita Banerjee et al. "New $\mathcal{N} = 2$ SuperBMS$_3$ algebra and invariant dual theory for 3D supergravity". In: *JHEP* 11 (2019), p. 122. DOI: 10.1007/JHEP11(2019)122. arXiv: 1905.10239 [hep-th].

[343] Nabamita Banerjee et al. "Free-field realisations of the BMS$_3$ algebra and its extensions". In: *JHEP* 06 (2016), p. 024. DOI: 10.1007/JHEP06(2016)024. arXiv: 1512.06240 [hep-th].

[344] Anders Schreiber, Anastasia Volovich, and Michael Zlotnikov. "Tree-level gluon amplitudes on the celestial sphere". In: *Phys. Lett. B* 781 (2018), pp. 349–357. DOI: 10.1016/j.physletb.2018.04.010. arXiv: 1711.08435 [hep-th].

[345] Sebastian Mizera and Sabrina Pasterski. "Celestial geometry". In: *JHEP* 09 (2022), p. 045. DOI: 10.1007/JHEP09(2022)045. arXiv: 2204.02505 [hep-th].

[346] C. N. Pope. "Lectures on W algebras and W gravity". In: *Summer School in High-energy Physics and Cosmology*. Dec. 1991, pp. 827–867. arXiv: hep-th/9112076.

[347] Adam Ball et al. "Perturbatively exact $\mathrm{w}_{1+\infty}$ asymptotic symmetry of quantum self-dual gravity". In: *JHEP* 01 (2022), p. 114. DOI: 10.1007/JHEP01(2022)114. arXiv: 2111.10392 [hep-th].

[348] Andrew Strominger. "$w_{1+\infty}$ Algebra and the Celestial Sphere: Infinite Towers of Soft Graviton, Photon, and Gluon Symmetries". In: *Phys. Rev. Lett.* 127.22 (2021), p. 221601. DOI: 10.1103/PhysRevLett.127.221601. arXiv: 2105.14346 [hep-th].

[349] Alfredo Guevara. "Notes on Conformal Soft Theorems and Recursion Relations in Gravity". In: (June 2019). arXiv: 1906.07810 [hep-th].

[350] Alfredo Guevara et al. "Holographic symmetry algebras for gauge theory and gravity". In: *JHEP* 11 (2021), p. 152. DOI: 10.1007/JHEP11(2021)152. arXiv: 2103.03961 [hep-th].

[351] Sabrina Pasterski and Shu-Heng Shao. "Conformal basis for flat space amplitudes". In: *Phys. Rev. D* 96.6 (2017), p. 065022. DOI: 10.1103/PhysRevD.96.065022. arXiv: 1705.01027 [hep-th].

[352] Angelos Fotopoulos and Tomasz R. Taylor. "Primary Fields in Celestial CFT". In: *JHEP* 10 (2019), p. 167. DOI: 10.1007/JHEP10(2019)167. arXiv: 1906.10149 [hep-th].

[353] Glenn Barnich and Pierre-Henry Lambert. "Einstein-Yang-Mills theory: Asymptotic symmetries". In: *Phys. Rev. D* 88 (2013), p. 103006. DOI: 10.1103/PhysRevD.88.103006. arXiv: 1310.2698 [hep-th].

[354] Tim Adamo, Lionel Mason, and Atul Sharma. "Celestial amplitudes and conformal soft theorems". In: *Class. Quant. Grav.* 36.20 (2019), p. 205018. DOI: 10.1088/1361-6382/ab42ce. arXiv: 1905.09224 [hep-th].

[355] A. Anastasiou et al. "Yang-Mills origin of gravitational symmetries". In: *Phys. Rev. Lett.* 113.23 (2014), p. 231606. DOI: 10.1103/PhysRevLett.113.231606. arXiv: 1408.4434 [hep-th].

[356] A. Anastasiou et al. "Gravity as Gauge Theory Squared: A Ghost Story". In: *Phys. Rev. Lett.* 121.21 (2018), p. 211601. DOI: 10.1103/PhysRevLett.121.211601. arXiv: 1807.02486 [hep-th].

[357] Gabriel Lopes Cardoso et al. "Comments on the double copy construction for gravitational theories". In: *PoS* CORFU2017 (2018), p. 177. DOI: 10.22323/1.318.0177. arXiv: 1803.07670 [hep-th].

[358] Andres Luna, Silvia Nagy, and Chris White. "The convolutional double copy: a case study with a point". In: *JHEP* 09 (2020), p. 062. DOI: 10.1007/JHEP09(2020)062. arXiv: 2004.11254 [hep-th].

[359] Leron Borsten et al. "Becchi-Rouet-Stora-Tyutin-Lagrangian Double Copy of Yang-Mills Theory". In: *Phys. Rev. Lett.* 126.19 (2021), p. 191601. DOI: 10.1103/PhysRevLett.126.191601. arXiv: 2007.13803 [hep-th].

[360] L. Borsten and S. Nagy. "The pure BRST Einstein-Hilbert Lagrangian from the double-copy to cubic order". In: *JHEP* 07 (2020), p. 093. DOI: 10.1007/JHEP07(2020)093. arXiv: 2004.14945 [hep-th].

[361] Walter D. Goldberger and Alexander K. Ridgway. "Radiation and the classical double copy for color charges". In: *Phys. Rev. D* 95.12 (2017), p. 125010. DOI: 10.1103/PhysRevD.95.125010. arXiv: 1611.03493 [hep-th].

[362] Chia-Hsien Shen. "Gravitational Radiation from Color-Kinematics Duality". In: *JHEP* 11 (2018), p. 162. DOI: 10.1007/JHEP11(2018)162. arXiv: 1806.07388 [hep-th].

[363] Walter D. Goldberger, Siddharth G. Prabhu, and Jedidiah O. Thompson. "Classical gluon and graviton radiation from the bi-adjoint scalar double copy". In: *Phys. Rev. D* 96.6 (2017), p. 065009. DOI: 10.1103/PhysRevD.96.065009. arXiv: 1705.09263 [hep-th].

[364] Walter D. Goldberger, Jingping Li, and Siddharth G. Prabhu. "Spinning particles, axion radiation, and the classical double copy". In: *Phys. Rev. D* 97.10 (2018), p. 105018. DOI: 10.1103/PhysRevD.97.105018. arXiv: 1712.09250 [hep-th].

[365] Walter D. Goldberger and Jingping Li. "Strings, extended objects, and the classical double copy". In: *JHEP* 02 (2020), p. 092. DOI: 10.1007/JHEP02(2020)092. arXiv: 1912.01650 [hep-th].

[366] Jan Plefka, Jan Steinhoff, and Wadim Wormsbecher. "Effective action of dilaton gravity as the classical double copy of Yang-Mills theory". In: *Phys. Rev. D* 99.2 (2019), p. 024021. DOI: 10.1103/PhysRevD.99.024021. arXiv: 1807.09859 [hep-th].

[367] Jan Plefka et al. "Breakdown of the classical double copy for the effective action of dilaton-gravity at NNLO". In: *Phys. Rev. D* 100.8 (2019), p. 086006. DOI: 10.1103/PhysRevD.100.086006. arXiv: 1906.05875 [hep-th].

[368] Jan Plefka, Canxin Shi, and Tianheng Wang. "Double copy of massive scalar QCD". In: *Phys. Rev. D* 101.6 (2020), p. 066004. DOI: 10.1103/PhysRevD.101.066004. arXiv: 1911.06785 [hep-th].

[369] Canxin Shi and Jan Plefka. "Classical double copy of worldline quantum field theory". In: *Phys. Rev. D* 105.2 (2022), p. 026007. DOI: 10.1103/PhysRevD.105.026007. arXiv: 2109.10345 [hep-th].

[370] Clifford Cheung and James Mangan. "Covariant color-kinematics duality". In: *JHEP* 11 (2021), p. 069. DOI: 10.1007/JHEP11(2021)069. arXiv: 2108.02276 [hep-th].

[371] Olaf Hohm. "On factorizations in perturbative quantum gravity". In: *JHEP* 04 (2011), p. 103. DOI: 10.1007/JHEP04(2011)103. arXiv: 1103.0032 [hep-th].

[372] Felipe Diaz-Jaramillo, Olaf Hohm, and Jan Plefka. "Double field theory as the double copy of Yang-Mills theory". In: *Phys. Rev. D* 105.4 (2022), p. 045012. DOI: 10.1103/PhysRevD.105.045012. arXiv: 2109.01153 [hep-th].

[373] Roberto Bonezzi, Felipe Diaz-Jaramillo, and Olaf Hohm. “The gauge structure of double field theory follows from Yang-Mills theory”. In: *Phys. Rev. D* 106.2 (2022), p. 026004. DOI: 10.1103/PhysRevD.106.026004. arXiv: 2203.07397 [hep-th].

[374] Clifford Cheung and Grant N. Remmen. “Twofold Symmetries of the Pure Gravity Action”. In: *JHEP* 01 (2017), p. 104. DOI: 10.1007/JHEP01(2017)104. arXiv: 1612.03927 [hep-th].

[375] Clifford Cheung and Grant N. Remmen. “Hidden Simplicity of the Gravity Action”. In: *JHEP* 09 (2017), p. 002. DOI: 10.1007/JHEP09(2017)002. arXiv: 1705.00626 [hep-th].

[376] Hans Stephani et al. *Exact solutions of Einstein’s field equations*. Cambridge University Press, 2003.

[377] Ricardo Monteiro, Donal O’Connell, and Chris D. White. “Black holes and the double copy”. In: *JHEP* 12 (2014), p. 056. DOI: 10.1007/JHEP12(2014)056. arXiv: 1410.0239 [hep-th].

[378] Gökhan Alka ç, Mehmet Kemal Gümü ş, and Mehmet Ali Olpak. “Kerr-Schild double copy of the Coulomb solution in three dimensions”. In: *Phys. Rev. D* 104 (4 Aug. 2021), p. 044034. DOI: 10.1103/PhysRevD.104.044034. URL: https://link.aps.org/doi/10.1103/PhysRevD.104.044034.

[379] Kanghoon Lee. “Kerr-Schild Double Field Theory and Classical Double Copy”. In: *JHEP* 10 (2018), p. 027. DOI: 10.1007/JHEP10(2018)027. arXiv: 1807.08443 [hep-th].

[380] Andrés Luna et al. “Type D Spacetimes and the Weyl Double Copy”. In: *Class. Quant. Grav.* 36 (2019), p. 065003. DOI: 10.1088/1361-6382/ab03e6. arXiv: 1810.08183 [hep-th].

[381] Damien A. Easson, Tucker Manton, and Andrew Svesko. “Einstein-Maxwell theory and the Weyl double copy”. In: *Phys. Rev. D* 107.4 (2023), p. 044063. DOI: 10.1103/PhysRevD.107.044063. arXiv: 2210.16339 [gr-qc].

[382] David A. Kosower, Ricardo Monteiro, and Donal O’Connell. “The SAGEX review on scattering amplitudes Chapter 14: Classical gravity from scattering amplitudes”. In: *J. Phys. A* 55.44 (2022), p. 443015. DOI: 10.1088/1751-8121/ac8846. arXiv: 2203.13025 [hep-th].

[383] Gang Chen and Yi-Jian Du. “Amplitude Relations in Non-linear Sigma Model”. In: *JHEP* 01 (2014), p. 061. DOI: 10.1007/JHEP01(2014)061. arXiv: 1311.1133 [hep-th].

[384] John Joseph M. Carrasco, Carlos R. Mafra, and Oliver Schlotterer. “Abelian Z-theory: NLSM amplitudes and α’-corrections from the open string”. In: *JHEP* 06 (2017), p. 093. DOI: 10.1007/JHEP06(2017)093. arXiv: 1608.02569 [hep-th].

[385] Freddy Cachazo, Song He, and Ellis Ye Yuan. “Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM”. In: *JHEP* 07 (2015), p. 149. DOI: 10.1007/JHEP07(2015)149. arXiv: 1412.3479 [hep-th].

[386] Z. Bern, C. Boucher-Veronneau, and H. Johansson. “N >= 4 Supergravity Amplitudes from Gauge Theory at One Loop”. In: *Phys. Rev. D* 84 (2011), p. 105035. DOI: 10.1103/PhysRevD.84.105035. arXiv: 1107.1935 [hep-th].

[387] Zvi Bern et al. “The Duality Between Color and Kinematics and its Applications”. In: (Sept. 2019). arXiv: 1909.01358 [hep-th].

[388] Ana-Maria Raclariu. "Lectures on Celestial Holography". In: (July 2021). arXiv: 2107.02075 [hep-th].

[389] Chi-Ming Chang et al. "Shadow celestial amplitudes". In: *Journal of High Energy Physics* 2023.2 (Feb. 2023), p. 17. ISSN: 1029-8479. DOI: 10.1007/JHEP02(2023)017. URL: https://doi.org/10.1007/JHEP02(2023)017.

[390] Nabamita Banerjee et al. "Supersymmetrization of deformed BMS algebras". In: (Jan. 2022). arXiv: 2201.09853 [hep-th].

[391] Eduardo Casali and Andrea Puhm. "Double Copy for Celestial Amplitudes". In: *Phys. Rev. Lett.* 126.10 (2021), p. 101602. DOI: 10.1103/PhysRevLett.126.101602. arXiv: 2007.15027 [hep-th].

[392] Nabamita Banerjee, Tabasum Rahnuma, and Ranveer Kumar Singh. "Asymptotic Symmetry algebra of $\mathcal{N} = 8$ Supergravity". In: (Dec. 2022). arXiv: 2212.12133 [hep-th].

[393] Wei Fan et al. "On Sugawara construction on Celestial Sphere". In: *JHEP* 09 (2020), p. 139. DOI: 10.1007/JHEP09(2020)139. arXiv: 2005.10666 [hep-th].

[394] Hongliang Jiang. "Holographic chiral algebra: supersymmetry, infinite Ward identities, and EFTs". In: *JHEP* 01 (2022), p. 113. DOI: 10.1007/JHEP01(2022)113. arXiv: 2108.08799 [hep-th].

[395] Sasha Haco et al. "Black Hole Entropy and Soft Hair". In: *JHEP* 12 (2018), p. 098. DOI: 10.1007/JHEP12(2018)098. arXiv: 1810.01847 [hep-th].

[396] Chong-Sun Chu and Yoji Koyama. "Soft Hair of Dynamical Black Hole and Hawking Radiation". In: *JHEP* 04 (2018), p. 056. DOI: 10.1007/JHEP04(2018)056. arXiv: 1801.03658 [hep-th].

[397] Artem Averin et al. "Goldstone origin of black hole hair from supertranslations and criticality". In: *Mod. Phys. Lett. A* 31.39 (2016), p. 1630045. DOI: 10.1142/S0217732316300457. arXiv: 1606.06260 [hep-th].

[398] Miguel S. Costa et al. "Spinning Conformal Correlators". In: *JHEP* 11 (2011), p. 071. DOI: 10.1007/JHEP11(2011)071. arXiv: 1107.3554 [hep-th].

[399] Paul A. M. Dirac. "Wave equations in conformal space". In: *Annals Math.* 37 (1936), pp. 429–442. DOI: 10.2307/1968455.

[400] Tim Adamo, Ricardo Monteiro, and Miguel F. Paulos. "Space-time CFTs from the Riemann sphere". In: *JHEP* 08 (2017), p. 067. DOI: 10.1007/JHEP08(2017)067. arXiv: 1703.04589 [hep-th].

[401] S. Ferrara, P. Gatto, and A. F. Grilla. "Conformal algebra in space-time and operator product expansion". In: *Springer Tracts Mod. Phys.* 67 (1973). Ed. by Dietrich Fries and Bernhard Zeitnitz, pp. 1–64. DOI: 10.1007/BFb0111104.

[402] G Mack and Abdus Salam. "Finite-component field representations of the conformal group". In: *Annals of Physics* 53.1 (1969), pp. 174–202. ISSN: 0003-4916. DOI: https://doi.org/10.1016/0003-4916(69)90278-4. URL: https://www.sciencedirect.com/science/article/pii/0003491669902784.

[403] Steven B. Giddings. "The Boundary S matrix and the AdS to CFT dictionary". In: *Phys. Rev. Lett.* 83 (1999), pp. 2707–2710. DOI: 10.1103/PhysRevLett.83.2707. arXiv: hep-th/9903048.

[404] Daniel Harlow and Douglas Stanford. “Operator Dictionaries and Wave Functions in AdS/CFT and dS/CFT”. In: (Apr. 2011). arXiv: 1104.2621 [hep-th].

[405] Daniel Kabat et al. “Holographic representation of bulk fields with spin in AdS/CFT”. In: *Phys. Rev. D* 86 (2012), p. 026004. DOI: 10.1103/PhysRevD.86.026004. arXiv: 1204.0126 [hep-th].

[406] Miguel S. Costa, Vasco Gonçalves, and João Penedones. “Spinning AdS Propagators”. In: *JHEP* 09 (2014), p. 064. DOI: 10.1007/JHEP09(2014)064. arXiv: 1404.5625 [hep-th].

[407] Bruce Allen and Theodore Jacobson. “Vector Two Point Functions in Maximally Symmetric Spaces”. In: *Commun. Math. Phys.* 103 (1986), p. 669. DOI: 10.1007/BF01211169.

[408] Asad Naqvi. “Propagators for massive symmetric tensor and p forms in AdS(d+1)”. In: *JHEP* 12 (1999), p. 025. DOI: 10.1088/1126-6708/1999/12/025. arXiv: hep-th/9911182.

[409] Nabamita Banerjee et al. “...work in progress”. In: (2024).

[410] Eliot Hijano et al. “Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks”. In: *JHEP* 01 (2016), p. 146. DOI: 10.1007/JHEP01(2016)146. arXiv: 1508.00501 [hep-th].

[411] Nabamita Banerjee et al. “Dual Theory for maximally $\mathcal{N}$ extended flat Supergravity”. In: (Oct. 2021). arXiv: 2110.05919 [hep-th].

[412] Glenn Barnich and Hernan A. Gonzalez. “Dual dynamics of three dimensional asymptotically flat Einstein gravity at null infinity”. In: *JHEP* 05 (2013), p. 016. DOI: 10.1007/JHEP05(2013)016. arXiv: 1303.1075 [hep-th].

[413] Glenn Barnich et al. “Super-BMS_3 invariant boundary theory from three-dimensional flat supergravity”. In: *JHEP* 01 (2017), p. 029. DOI: 10.1007/JHEP01(2017)029. arXiv: 1510.08824 [hep-th].

[414] Ricardo Caroca et al. “Three-dimensional Poincaré supergravity and N-extended supersymmetric BMS_3 algebra”. In: *Phys. Lett. B* 792 (2019), pp. 93–100. DOI: 10.1016/j.physletb.2019.02.049. arXiv: 1812.05065 [hep-th].

[415] Nabamita Banerjee, Saurish Khandelwal, and Parita Shah. “Equivalent dual theories for 3D $\mathcal{N}=2$ supergravity”. In: *Phys. Rev. D* 100.10 (2019), p. 105013. DOI: 10.1103/PhysRevD.100.105013. arXiv: 1907.05866 [hep-th].

[416] Artem Averin et al. “Gravitational Black Hole Hair from Event Horizon Supertranslations”. In: *JHEP* 06 (2016), p. 088. DOI: 10.1007/JHEP06(2016)088. arXiv: 1601.03725 [hep-th].

[417] Laura Donnay et al. “Supertranslations and Superrotations at the Black Hole Horizon”. In: *Phys. Rev. Lett.* 116.9 (2016), p. 091101. DOI: 10.1103/PhysRevLett.116.091101. arXiv: 1511.08687 [hep-th].

[418] Mehrdad Mirbabayi and Massimo Porrati. “Dressed Hard States and Black Hole Soft Hair”. In: *Phys. Rev. Lett.* 117.21 (2016), p. 211301. DOI: 10.1103/PhysRevLett.117.211301. arXiv: 1607.03120 [hep-th].

[419] Raphael Bousso and Massimo Porrati. “Soft Hair as a Soft Wig”. In: *Class. Quant. Grav.* 34.20 (2017), p. 204001. DOI: 10.1088/1361-6382/aa8be2. arXiv: 1706.00436 [hep-th].

[420] Qu Cao et al. “Supergluon scattering in AdS: constructibility, spinning amplitudes, and new structures”. In: (June 2024). arXiv: 2406.08538 [hep-th].

[421] A. Zee. *Einstein Gravity in a Nutshell*. New Jersey: Princeton University Press, May 2013. ISBN: 978-0-691-14558-7.

Appendices

APPENDIX A
SOME REQUIRED INTEGRALS

A.1 Evaluation of Useful Integrals

We show that certain integrals vanish as a result of the global conformal invariance of celestial correlators. These results are used in the computations presented in the main draft.

A.1.1 Evaluation of Integral I

We have,

$$\left\langle G^a(z)G^b(w)\prod_{n=2}^{N}\mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = -\frac{1}{2\pi}\sum_{i=1}^{M}\sum_{c} f^{ab_ic}\int d^2z_1\frac{1}{(w-z_1)^2}\frac{1}{z-z_i} \times\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$

Now using the global conformal invariance of $\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$, we will show that the first integral vanishes and the only nontrivial terms which exist are the regular ones.

Putting $W = w - z_1$, $Z = z - z_1$ we have,

$$\int d^2z_1\,\frac{1}{W^2Z}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = -\partial_w\int d^2z_1\,\frac{1}{WZ}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$

We use the following identity:

$$\frac{1}{WZ} = \frac{1}{z-w}\left(\frac{1}{w-z_1}-\frac{1}{z-z_1}\right) = \frac{1}{z-w}\left(\frac{1}{W}-\frac{1}{Z}\right)$$

We get,

$$\int d^2z_1\,\frac{1}{WZ}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \frac{1}{z-w}\int d^2z_1\left(\frac{1}{W}-\frac{1}{Z}\right)\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle.$$

We change $z_1 \to z - w + z_1$ in the second term of this integral:

$$\int d^2z_1\,\frac{1}{z-z_1}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \longrightarrow \int d^2z_1\,\frac{1}{w-z_1}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$

By global conformal invariance of

$$\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\ldots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle$$

under $\begin{pmatrix} 1 & z-w \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, we get (cf. [185, Eq. (3.2)]),

$$\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z-w+z_1,\bar{z}-\bar{w}+\bar{z}_1)\dots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\dots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = \left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\dots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\dots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle,$$

which implies

$$\int d^2z_1\,\frac{1}{WZ}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\dots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\dots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = 0$$

Thus we get

$$\int d^2z_1\,\frac{1}{W^2Z}\left\langle \mathcal{O}^{b_1=b}_{1,-1}(z_1,\bar{z}_1)\dots\mathcal{O}^{c}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\dots\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = -\partial_w(0) = 0$$

A.1.2 Evaluation of Integral II

We show that

$$\frac{1}{2\pi}\int d^2z_1\left(\frac{1}{z-w}\frac{1}{WZ}-\frac{1}{WZ^2}\right)\left\langle \mathcal{O}^{a}_{1,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = 0.$$

It suffices to prove that

$$\frac{1}{2\pi}\int d^2z_1\frac{1}{WZ}\left\langle \mathcal{O}^{a}_{1,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle = 0,$$

since the second term is given by

$$\begin{aligned}\frac{1}{2\pi}\int d^2z_1\frac{1}{WZ^2}&\left\langle \mathcal{O}^{a}_{1,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\ &= -\frac{1}{2\pi}\partial_z\left[\int d^2z_1\frac{1}{WZ}\left\langle \mathcal{O}^{a}_{1,-1}(z_1,\bar{z}_1)\prod_{n=2}^{M}\mathcal{O}^{b_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\right] \\ &= 0\end{aligned}$$

Using the global conformal invariance of the correlator as in A.1.1, the proof is complete.

A.2 Conformal Dimension of $\mathcal{G}^{ab}(z,\bar{z})$

In this appendix, we explicitly compute the OPE of $\mathcal{G}^{ab}(z,\bar{z})$ with $T(z)$ and $\overline{T}(\bar{z})$ and conclude that the conformal dimension of $\mathcal{G}^{ab}(z,\bar{z})$ is $(h,\bar{h})=(1,1)$. This justifies the mode expansion of $\mathcal{G}^{ab}(z,\bar{z})$ in Eq. (5.88). We have

$$\left\langle T(z)\mathcal{G}^{ab}(w,\bar{w})\right\rangle = \left\langle T(z)\ :G^a(w)\overline{G}^b(\bar{w}):\ \right\rangle - \left\langle T(z)\ :\overline{G}^b(\bar{w})G^a(w):\ \right\rangle$$

which is the difference of two three point correlation functions. We can use generalised Wick's theorem (see [58, Appendix 6.B]) to simplify this. Since the normal ordering ': :' removes the singular terms from the OPE, we can use the expression [58, Appendix B, Eq. (6.206)]. Then we have

$$\begin{aligned}
&\Big\langle T(z)\ :G^a(w)\overline{G}^b(\bar{w}):\ \Big\rangle \\
&= \frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\overbrace{T(z)\ :G^a(x)}\overline{G}^b(\bar{w}): + :\ G^a(x)\overbrace{T(z)\ \overline{G}^b(\bar{w})}:\right] \\
&= \frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\frac{1}{(z-x)^2}:G^a(x)\overline{G}^b(\bar{w}): + \frac{1}{z-x}\ :\partial_x G^a(x)\overline{G}^b(\bar{w}):\ +\ \text{regular.}\right] \\
&= \frac{1}{(z-w)^2}:G^a(w)\overline{G}^b(\bar{w}): + \frac{1}{z-w}:\partial_w G^a(w)\overline{G}^b(\bar{w}):,
\end{aligned}$$

where we used the OPE of Eq. (3.28) and Eq. (3.31). Similarly

$$\begin{aligned}
&\Big\langle T(z)\ :\overline{G}^b(\bar{w})G^a(w):\ \Big\rangle \\
&= \frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\overbrace{T(z)\ :\overline{G}^b(}\bar{w})G^a(x): + :\overline{G}^b(\bar{w})\overbrace{T(z)\ G^a(}x):\right] \\
&= \frac{1}{2\pi i}\oint \frac{dx}{x-w}\left[\text{regular.} + \frac{1}{(z-x)^2}:\overline{G}^b(\bar{w})G^a(x): + \frac{1}{z-x}\ :\overline{G}^b(\bar{w})\partial_x G^a(x):\right] \\
&= \frac{1}{(z-w)^2}:\overline{G}^b(\bar{w})G^a(w): + \frac{1}{z-w}:\overline{G}^b(\bar{w})\partial_w G^a(w):,
\end{aligned}$$

where we again used the OPE of Eq. (3.28). Thus we get

$$\begin{aligned}
\Big\langle T(z)\mathcal{G}^{ab}(w,\bar{w})\Big\rangle &= \frac{1}{(z-w)^2}:\Big[G^a(w),\overline{G}^b(\bar{w})\Big]: + \frac{1}{z-w}\partial_w:\Big[G^a(w),\overline{G}^b(\bar{w})\Big]: \\
&= \frac{1}{(z-w)^2}\mathcal{G}^{ab}(w,\bar{w}) + \frac{1}{z-w}\partial_w\mathcal{G}^{ab}(w,\bar{w}).
\end{aligned}$$

Similarly we can compute the OPE of $\mathcal{G}^{ab}(z,\bar{z})$ with $\overline{T}(\bar{z})$. We get

$$\Big\langle \overline{T}(\bar{z})\mathcal{G}^{ab}(w,\bar{w})\Big\rangle = \frac{1}{(\bar{z}-\bar{w})^2}\mathcal{G}^{ab}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\mathcal{G}^{ab}(w,\bar{w}).$$

APPENDIX B
ALL COLLINEAR SPLIT FACTORS

B.1 Split Factors

Here, we list all of the split factors corresponding to both like and unlike spins in our supergravity theory.
Gravi-photons splits:

$$
\begin{aligned}
&\mathrm{Split}_0^{\mathrm{SG}}\left(z,1^{1+0},2^{1+0}\right)=-\frac{[12]}{\langle 12\rangle}, \qquad \mathrm{Split}_0^{\mathrm{SG}}\left(z,1^{1+0},2^{0+1}\right)=-\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}_0^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{\frac{1}{2}+\frac{1}{2}}\right)=-\frac{[12]}{\langle 12\rangle}, \qquad \mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{1+0},2^{-1+0}\right)=-(1-z)^2\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{1+0},2^{-1+0}\right)=-z^2\frac{\langle 12\rangle}{[12]}, \qquad \mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{\frac{1}{2}+\frac{1}{2}}\right)=-(1-z)^2\frac{\langle 12\rangle}{[12]}\\
&\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{\frac{1}{2}+\frac{1}{2}}\right)=-z^2\frac{[12]}{\langle 12\rangle}
\end{aligned}
\tag{B.1}
$$

Gravi-photinos splits:

$$
\begin{aligned}
&\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{\frac{1}{2}+0}\right)=-\sqrt{z(1-z)}\frac{[12]}{\langle 12\rangle}, \qquad \mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{0+\frac{1}{2}}\right)=-\sqrt{z(1-z)}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{-\frac{1}{2}+0}\right)=-\sqrt{z(1-z)^3}\frac{[12]}{\langle 12\rangle}, \qquad \mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{-\frac{1}{2}+0}\right)=-\sqrt{z^3(1-z)}\frac{\langle 12\rangle}{[12]}\\
&\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{1-\frac{1}{2}},2^{-\frac{1}{2}+1}\right)=-\sqrt{z(1-z)}\frac{[12]}{\langle 12\rangle}, \qquad \mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}-1},2^{-\frac{1}{2}+1}\right)=-\sqrt{z(1-z)^3}\frac{\langle 12\rangle}{[12]}\\
&\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}-1},2^{-\frac{1}{2}+1}\right)=-\sqrt{z^3(1-z)}\frac{[12]}{\langle 12\rangle}.
\end{aligned}
\tag{B.2}
$$

Scalars Splits:

$$
\begin{aligned}
&\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{0+0},2^{0+0}\right)-z(1-z)\frac{\langle 12\rangle}{[12]}, \qquad \mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{0+0},2^{0+0}\right)=-z(1-z)\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{-1+1},2^{+1-1}\right)=-z(1-z)\frac{\langle 12\rangle}{[12]}, \qquad \mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{-1+1},2^{+1-1}\right)=-z(1-z)\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}+\frac{1}{2}},2^{+\frac{1}{2}-\frac{1}{2}}\right)=-z(1-z)\frac{\langle 12\rangle}{[12]}, \qquad \mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}+\frac{1}{2}},2^{+\frac{1}{2}-\frac{1}{2}}\right)=-z(1-z)\frac{[12]}{\langle 12\rangle}.
\end{aligned}
\tag{B.3}
$$

Graviton-Gravitino Splits:

$$
\begin{aligned}
&\mathrm{Split}_{-\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{1+1},2^{1+\frac{1}{2}}\right)=-\frac{1}{z\sqrt{1-z}}\frac{[12]}{\langle 12\rangle}, \qquad \mathrm{Split}_{-\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{1+1},2^{\frac{1}{2}+1}\right)=-\frac{1}{z\sqrt{(1-z)}}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}_{+\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{1+1},2^{-1-\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)^5}}{z}\frac{[12]}{\langle 12\rangle}, \qquad \mathrm{Split}_{+\frac{3}{2}}^{\mathrm{SG}}\left(z,1^{1+1},2^{-\frac{1}{2}-1}\right)=-\frac{\sqrt{(1-z)^5}}{z}\frac{[12]}{\langle 12\rangle}
\end{aligned}
\tag{B.4}
$$

All other Splits can be written via the helicity flipping relation in Eq. (4.25).

Graviton-Graviphoton splits:

$$\text{Split}^{\text{SG}}_{-1}\left(z,1^{1+1},2^{1+0}\right)=-\frac{1}{z}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{-1}\left(z,1^{1+1},2^{0+1}\right)=-\frac{1}{z}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{+1}\left(z,1^{1+1},2^{0-1}\right)=-\frac{(1-z)^2}{z}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{+1}\left(z,1^{1+1},2^{-1+0}\right)=-\frac{(1-z)^2}{z}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{-1}\left(z,1^{1+1},2^{\frac{1}{2}+\frac{1}{2}}\right)=-\frac{1}{z}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{+1}\left(z,1^{1+1},2^{-\frac{1}{2}-\frac{1}{2}}\right)=-\frac{(1-z)^2}{z}\frac{[12]}{\langle 12\rangle} \tag{B.5}$$

Rest are summarised in Eq. (4.25).

Graviton-Graviphotino splits:

$$\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{\frac{1}{2}+0}\right)=-\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{-\frac{1}{2}+0}\right)=-\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{0+\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{0-\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{1-\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{-1+\frac{1}{2}}\right)=-\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{-\frac{1}{2}+1}\right)=-\frac{\sqrt{(1-z)}}{z}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{\frac{1}{2}-1}\right)=-\frac{\sqrt{(1-z)^3}}{z}\frac{[12]}{\langle 12\rangle} \tag{B.6}$$

Graviton-Scalar Splits:

$$\begin{aligned}\text{Split}^{\text{SG}}_{0}\left(z,1^{1+1},2^{0+0}\right)&=-\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}\\ \text{Split}^{\text{SG}}_{0}\left(z,1^{1+1},2^{1-1}\right)&=-\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}\\ \text{Split}^{\text{SG}}_{0}\left(z,1^{1+1},2^{\frac{1}{2}-\frac{1}{2}}\right)&=-\frac{(1-z)}{z}\frac{[12]}{\langle 12\rangle}\end{aligned} \tag{B.7}$$

Gravitino-Graviphoton Splits:

$$\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{0+1}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{1+0}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+\frac{1}{2}}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+\frac{1}{2}},2^{0+1}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+\frac{1}{2}},2^{1+0}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+\frac{1}{2}},2^{\frac{1}{2}+\frac{1}{2}}\right)=-\frac{1}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$
$$\text{Split}^{\text{SG}}_{+\frac{3}{2}}\left(z,1^{\frac{1}{2}+1},2^{0-1}\right)=-\frac{(1-z)^2}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\qquad \text{Split}^{\text{SG}}_{+\frac{3}{2}}\left(z,1^{1+\frac{1}{2}},2^{-1+0}\right)=-\frac{(1-z)^2}{\sqrt{z}}\frac{[12]}{\langle 12\rangle} \tag{B.8}$$

Gravitino-Graviphotino Splits:

$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{0+\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle}$$
$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{1-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}+1}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle}$$
$$\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{\frac{1}{2}+1},2^{0-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle 12\rangle}$$
$$\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}-1}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{0+\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle}$$
$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{1-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle}$$
$$\mathrm{Split}^{\mathrm{SG}}_{0}\left(z,1^{1+\frac{1}{2}},2^{-\frac{1}{2}+1}\right)=-\sqrt{\frac{(1-z)}{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+\frac{1}{2}},2^{0-\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle 12\rangle}$$
$$\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+\frac{1}{2}},2^{-\frac{1}{2}+0}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+\frac{1}{2}},2^{-1+\frac{1}{2}}\right)=-\sqrt{\frac{(1-z)^3}{z}}\frac{[12]}{\langle 12\rangle} \tag{B.9}$$

Gravitino-Scalar Splits:

$$\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{0+0}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{1-1}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle 12\rangle}$$
$$\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}-\frac{1}{2}}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle 12\rangle},\qquad \mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{1+\frac{1}{2}},2^{0+0}\right)=-\frac{(1-z)}{\sqrt{z}}\frac{[12]}{\langle 12\rangle} \tag{B.10}$$

Similarly for other factorisations of Gravitino we have the same split factors.
Graviphoton-Graviphotino Splits:

$$
\begin{aligned}
&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{0+1},2^{0+\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{0+1},2^{\frac{1}{2}+0}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{0+1},2^{1-\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0+\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{\frac{1}{2}+0}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{1-\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}+1}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{1+0},2^{0+\frac{1}{2}}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{1+0},2^{\frac{1}{2}+0}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{1+0},2^{-\frac{1}{2}+1}\right)=-\sqrt{(1-z)}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+1},2^{0-\frac{1}{2}}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+1},2^{\frac{1}{2}-1}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0-\frac{1}{2}}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}+0}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{1+0},2^{-\frac{1}{2}+0}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{1+0},2^{-1+\frac{1}{2}}\right)=-(1-z)^{\frac{3}{2}}\frac{[12]}{\langle 12\rangle}
\end{aligned}
\tag{B.11}
$$

Graviphoton-Scalar Splits:

$$
\begin{aligned}
&\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{0+1},2^{0+0}\right)=-(1-z)\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{0+1},2^{1-1}\right)=-(1-z)\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{0+1},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{1+0},2^{0+0}\right)=-(1-z)\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0+0}\right)=-(1-z)\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{-1+0},2^{1-1}\right)=-(1-z)\frac{\langle 12\rangle}{[12]}, &&\mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{-1+0},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{\langle 12\rangle}{[12]}\\
&\mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-(1-z)\frac{\langle 12\rangle}{[12]}
\end{aligned}
\tag{B.12}
$$

Graviphotino-Scalar Splits:

$$
\begin{aligned}
&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+\frac{1}{2}},2^{0+0}\right)=-z^{\frac{1}{2}}(1-z)\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-z^{\frac{1}{2}}(1-z)\frac{[12]}{\langle 12\rangle}\\
&\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{\frac{1}{2}+0},2^{0+0}\right)=-z^{\frac{1}{2}}(1-z)\frac{[12]}{\langle 12\rangle}, &&\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-\frac{1}{2}+0},2^{\frac{1}{2}-\frac{1}{2}}\right)=-z^{\frac{1}{2}}(1-z)\frac{\langle 12\rangle}{[12]}\\
&\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-1+\frac{1}{2}},2^{1-1}\right)=-z^{\frac{1}{2}}(1-z)\frac{\langle 12\rangle}{[12]}, &&\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-1+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)=-z^{\frac{1}{2}}(1-z)\frac{\langle 12\rangle}{[12]}
\end{aligned}
\tag{B.13}
$$

B.2 Explicit computations of Amplitudes

In this appendix, we explicitly calculate the collinear limits of states various spin combinations.

B.2.1 Like spins

The collinear limits of gravitons is calculated in Section B.2.1 in detail. So we start with collinear limit of gravitinos.

Gravitinos
The factorisation of R-symmetry indices has the form

$$\begin{cases} \left(a;\frac{3}{2}\right) = \left(a;\frac{1}{2}\right)\otimes 1 \\ \left(r;\frac{3}{2}\right) = 1\otimes\left(r;\frac{1}{2}\right). \end{cases}$$

We then have

$$\begin{aligned} M_n\left(1^{a;+\frac{3}{2}},2^{b;+\frac{3}{2}},\cdots,n\right) &= M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{\left(b;\frac{1}{2}\right)\otimes 1},\cdots,n\right) \\ &= \text{Split}^{\text{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+1}\right)\times M_{n-1}\left(p^{ab;+1},\cdots,n\right) \\ &= \frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{ab;+1},\cdots,n\right) \end{aligned} \tag{B.14}$$

$$\begin{aligned} M_n\left(1^{a;+\frac{3}{2}},2^{r;+\frac{3}{2}},\cdots,n\right) &= M_n\left(1^{\left(a;\frac{1}{2}\right)\otimes 1},2^{1\otimes\left(r;\frac{1}{2}\right)},\cdots,n\right) \\ &= \text{Split}^{\text{SG}}_{-1}\left(z,1^{\frac{1}{2}+1},2^{1+\frac{1}{2}}\right)\times M_{n-1}\left(p^{ar;+1},\cdots,n\right) \\ &= \frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{ar;+1},\cdots,n\right) \end{aligned} \tag{B.15}$$

The collinear limits remains the same under $(a,b)\to(r,s)$. All these can be combined and we can write

$$M_n\left(1^{A;+\frac{3}{2}},2^{B;+\frac{3}{2}},\cdots,n\right) = \frac{\omega_p}{\sqrt{\omega_1\omega_2}}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{AB;+1},\cdots,n\right) \tag{B.16}$$

For opposite helicities, we have

$$\begin{aligned} M_n\left(1^{a;+\frac{3}{2}},2_b^{-\frac{3}{2}},\cdots,n\right) &= \delta^a_b\text{Split}^{\text{SG}}_{-2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right)M_{n-1}\left(p^{+2},\cdots,n\right) \\ &\quad+\delta^a_b\text{Split}^{\text{SG}}_{+2}\left(z,1^{\frac{1}{2}+1},2^{-\frac{1}{2}-1}\right)M_{n-1}\left(p^{-2},\cdots,n\right) \\ &= \delta^a_b\frac{\omega_2^{\frac{5}{2}}}{\omega_1^{\frac{1}{2}}\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}M_{n-1}\left(p^{-2},\cdots,n\right)+\delta^a_b\frac{\omega_1^{\frac{5}{2}}}{\omega_2^{\frac{1}{2}}\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}M_{n-1}\left(p^{+2},\cdots,n\right). \end{aligned} \tag{B.17}$$

The collinear limit remains the same under $(a,b)\to(r,s)$. Infact since there are no other nontrivial split factors for other factorisations, the above collinear limit is true for any $1\leq$

$A, B \leq 8$:

$$M_n\left(1^{A;+\frac{3}{2}}, 2_B^{-\frac{3}{2}}, \cdots, n\right) = \delta_B^A \frac{\omega_2^{\frac{5}{2}}}{\omega_1^{\frac{1}{2}}\omega_p^2} \frac{\bar{z}_{12}}{z_{12}} M_{n-1}\left(p^{-2}, \cdots, n\right) + \delta_B^A \frac{\omega_1^{\frac{5}{2}}}{\omega_2^{\frac{1}{2}}\omega_p^2} \frac{z_{12}}{\bar{z}_{12}} M_{n-1}\left(p^{+2}, \cdots, n\right). \tag{B.18}$$

Graviphotons
The factorizations of R-symmetry indices are as follows,

$$\begin{cases} (ab;1) = (ab;0) \otimes 1 \\ (ar;1) = \left(a, \frac{1}{2}\right) \otimes \left(r; \frac{1}{2}\right) \\ (rs;1) = 1 \otimes (rs;0) \end{cases}$$

Using the split factors from Appendix B.1 we have

$$\begin{aligned} M_n\left(1^{ab;+1}, 2^{cd;+1}, \cdots, n\right) &= M_n\left(1^{ab;(1\otimes 0)}, 2^{cd;(1\otimes 0)}, \cdots, n\right) \\ &= \text{Split}_0^{\text{SG}}\left(z, 1^{1+0}, 2^{1+0}\right) \times M_{n-1}\left(p^{abcd;0}, \cdots, n\right) \\ &= \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{abcd;0}, \cdots, n\right) \end{aligned} \tag{B.19}$$

Similarly

$$M_n\left(1^{rs;+1}, 2^{tu;+1}, \cdots, n\right) = \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{rstu;0}, \cdots, n\right)$$

Next

$$\begin{aligned} M_n\left(1^{rs;+1}, 2^{ab;+1}, \cdots, n\right) &= M_n\left(1^{rs;(1\otimes 0)}, 2^{ab;(0\otimes 1)}, \cdots, n\right) \\ &= \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{rsab;0}, \cdots, n\right) \end{aligned} \tag{B.20}$$

Similarly

$$M_n\left(1^{ab;+1}, 2^{rs;+1}, \cdots, n\right) = \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{abrs;0}, \cdots, n\right)$$

Next

$$\begin{aligned} M_n\left(1^{ar;+1}, 2^{bs;+1}, \cdots, n\right) &= M_n\left(1^{ar;(\frac{1}{2}\otimes\frac{1}{2})}, 2^{bs;(\frac{1}{2}\otimes\frac{1}{2})}, \cdots, n\right) \\ &= \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{arbs;0}, \cdots, n\right) \end{aligned} \tag{B.21}$$

These can be combined to write the collinear limit uniformly as

$$M_n\left(1^{AB;+1}, 2^{CD;+1}, \cdots, n\right) = \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{ABCD;0}, \cdots, n\right) \tag{B.22}$$

where $1 \leq A, B \leq 8$. For opposite helicities we have,

$$\begin{aligned} M_n\left(1^{ar;+1}, 2^{-1}_{bs}, \cdots, n\right) &= M_n\left(1^{ar;(\frac{1}{2}\otimes\frac{1}{2})}, 2^{(-\frac{1}{2}\otimes-\frac{1}{2})}_{bs}, \cdots, n\right) \\ &= -\delta^a_b \delta^r_s \Big[\text{Split}^{\text{SG}}_{-2}\left(z, 1^{\frac{1}{2}+\frac{1}{2}}, 2^{-\frac{1}{2}-\frac{1}{2}}\right) \times M_{n-1}\left(p^{+2}, \cdots, n\right) \\ &\quad + \text{Split}^{\text{SG}}_{+2}\left(z, 1^{\frac{1}{2}+\frac{1}{2}}, 2^{-\frac{1}{2}-\frac{1}{2}}\right) \times M_{n-1}\left(p^{-2}, \cdots, n\right)\Big] \\ &= -\delta^a_b \delta^r_s \left[\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots, n\right) + \frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots, n\right)\right] \end{aligned} \tag{B.23}$$

Note that the negative sign in the first comes from the negative sign in the factorisation of negative helicity graviphotons.
Similarly,

$$\begin{aligned} M_n\left(1^{ab;+1}, 2^{-1}_{cd}, \cdots, n\right) &= M_n\left(1^{ab;(1\otimes 0)}, 2^{(-1\otimes 0)}_{cd}, \cdots, n\right) \\ &= \frac{1}{2!}\alpha_4 \varepsilon_{cdef}\varepsilon^{abef}\Big[\text{Split}^{\text{SG}}_{-2}\left(z, 1^{1+0}, 2^{-1+0}\right) \times M_{n-1}\left(p^{+2}, \cdots, n\right) \\ &\quad + \text{Split}^{\text{SG}}_{+2}\left(z, 1^{1+0}, 2^{-1+0}\right) \times M_{n-1}\left(p^{-2}, \cdots, n\right)\Big] \\ &= \alpha_4 \delta^{ab}_{cd}\left[\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots, n\right) + \frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots, n\right)\right] \end{aligned} \tag{B.24}$$

where the generalised Kronecker delta $\delta^{a_1 \dots a_n}_{b_1 \dots b_n}$ is defined as

$$\delta^{a_1 \dots a_n}_{b_1 \dots b_n} = \sum_{\sigma \in S_n} \text{sign}(\sigma)\delta^{a_{\sigma(1)}}_{b_1} \dots \delta^{a_{\sigma(n)}}_{b_n}, \tag{B.25}$$

and we used the self-duality condition Eq.(4.21). This collinear limit remains the same under $(a,b,c,d) \to (r,s,t,u)$ with α_4 replaced by $\tilde{\alpha}_4$. Thus if we pick $\alpha_4 = \tilde{\alpha}_4 = -1$, then using the fact that $\delta^a_r = 0$, we can write the collinear limit of two opposite helicity gauge bosons collectively as

$$\begin{aligned} M_n\left(1^{AB;+1}, 2^{-1}_{CD}, \cdots, n\right) = -\delta^{AB}_{CD}\Big[&\frac{\omega_2^2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots, n\right) \\ &+ \frac{\omega_1^2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots, n\right)\Big]. \end{aligned} \tag{B.26}$$

Our choice of the parameters α_4 and $\tilde{\alpha}_4$ is purely motivated by our desire to combine the collinear limits for different factorisations of the gauge bosons in supergravity. Other choices of the parameters will introduce some extra negative signs in some of the collinear limits.

Graviphotinos

The factorisation R-symmetry indices is given by

$$\begin{cases} \left(abr;\tfrac{1}{2}\right) = (ab;0)\otimes\left(r;\tfrac{1}{2}\right) \\ \left(ars;\tfrac{1}{2}\right) = \left(a;\tfrac{1}{2}\right)\otimes(rs;0) \end{cases}$$

$$\begin{cases} \left(rst;\tfrac{1}{2}\right) = -\varepsilon^{rstu}(1\otimes(u;-\tfrac{1}{2})) \\ \left(abc;\tfrac{1}{2}\right) = -\varepsilon^{abcd}((d;-\tfrac{1}{2})\otimes 1) \end{cases} \qquad \text{(sum over } u,d\text{)}$$

Using this factoriation and the split factors in Appendix B.1 the collinear limits of various combination of R-symmetry indices is calculated below. We have

$$\begin{aligned} M_n\left(1^{ars;+\frac{1}{2}},2^{btu;\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{ars;(\frac{1}{2}\otimes 0)},2^{btu;(\frac{1}{2}\otimes 0)},\cdots,n\right) \\ &= \varepsilon^{rstu}\varepsilon^{abcd}\,\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{\frac{1}{2}+0}\right)\times M_{n-1}\left(p_{cd}^{-1},\cdots,n\right) \\ &= \varepsilon^{rstu}\varepsilon^{abcd}\,\frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{cd}^{-1},\cdots,n\right). \end{aligned} \tag{B.27}$$

Here the ε^{rstu} factor appears because of the collinear split factor between two scalars in $\mathcal{N}=4$ SYM.
Similarly for other non-trivial factorisation we have,

$$\begin{aligned} M_n\left(1^{ars;+\frac{1}{2}},2^{bct;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{ars;(\frac{1}{2}\otimes 0)},2^{bct;(0\otimes\frac{1}{2})},\cdots,n\right) \\ &= \varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{du}^{-1},\cdots,n\right) \end{aligned} \tag{B.28}$$

$$\begin{aligned} M_n\left(1^{rst;+\frac{1}{2}},2^{abc;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{rst;(1\otimes-\frac{1}{2})},2^{abc;(-\frac{1}{2}\otimes 1)},\cdots,n\right) \\ &= \varepsilon^{rstu}\varepsilon^{abcd}\,\frac{\sqrt{\omega_1\omega_2}}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{ud}^{-1},\cdots,n\right) \end{aligned} \tag{B.29}$$

$$\begin{aligned} M_n\left(1^{ars;+\frac{1}{2}},2_{btu}^{-\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{ars;(\frac{1}{2}\otimes 0)},2_{btu}^{(-\frac{1}{2}\otimes 0)},\cdots,n\right) \\ &= \varepsilon_{tuvw}\varepsilon^{rsvw}\delta_b^a\Big[\mathrm{Split}_{-2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{-\frac{1}{2}+0}\right)\times M_{n-1}\left(p^{+2},\cdots,n\right) \\ &\quad+\mathrm{Split}_{+2}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+0},2^{-\frac{1}{2}+0}\right)\times M_{n-1}\left(p^{-2},\cdots,n\right)\Big] \\ &= \varepsilon_{tuvw}\varepsilon^{rsvw}\delta_b^a\Big[\frac{\omega_1^{\frac{3}{2}}\omega_2^{\frac{1}{2}}}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{+2},\cdots,n\right) \\ &\quad+\frac{\omega_2^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-2},\cdots,n\right)\Big] \end{aligned} \tag{B.30}$$

Scalars
The three possible channels are $0=0\otimes 0$, $0=\pm 1\otimes\mp 1$ and $0=\pm\frac{1}{2}\otimes\mp\frac{1}{2}$. We have the

non-trivial splits are given in Appendix B.3. The factorization of R-symmetry indices are,

$$(abrs;0) = (ab;0) \otimes (rs;0)$$

$$\begin{cases} (abcd;0) = -\varepsilon^{abcd}(-1 \otimes 1) \\ (rstu;0) = -\varepsilon^{rstu}(1 \otimes -1) \end{cases}$$

$$\begin{cases} (abcr;0) = -\varepsilon^{abcd}\left(d;-\tfrac{1}{2}\right) \otimes \left(r;\tfrac{1}{2}\right) \\ (arst;0) = -\varepsilon^{rstu}\left(a;\tfrac{1}{2}\right) \otimes \left(u;-\tfrac{1}{2}\right) \end{cases}$$

The collinear amplitudes are then given by

$$\begin{aligned} M_n\left(1^{abrs;0}, 2^{cdtu;0}, \cdots, n\right) &= M_n\left(1^{abrs;(0\otimes 0)}, 2^{cdtu;(0\otimes 0)}, \cdots, n\right) \\ &= \varepsilon^{abcd}\varepsilon^{rstu}\Big[\mathrm{Split}^{\mathrm{SG}}_{-2}\left(z, 1^{0+0}, 2^{0+0}\right) \times M_{n-1}\left(p^{+2}, \cdots, n\right) \\ &\quad + \mathrm{Split}^{\mathrm{SG}}_{+2}\left(z, 1^{0+0}, 2^{0+0}\right) \times M_{n-1}\left(p^{-2}, \cdots, n\right)\Big] \\ &= \varepsilon^{abcd}\varepsilon^{rstu}\left[\frac{\omega_1\omega_2}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots, n\right)\right. \\ &\quad \left. + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots, n\right)\right] \end{aligned} \tag{B.31}$$

$$\begin{aligned} M_n\left(1^{abcd;0}, 2^{rstu;0}, \cdots, n\right) &= M_n\left(1^{abcd;(-1\otimes 1)}, 2^{rstu;(+1\otimes -1)}, \cdots, n\right) \\ &= \varepsilon^{abcd}\varepsilon^{rstu}\Big[\mathrm{Split}^{\mathrm{SG}}_{-2}\left(z, 1^{-1+1}, 2^{+1-1}\right) \times M_{n-1}\left(p^{+2}, \cdots, n\right) \\ &\quad + \mathrm{Split}^{\mathrm{SG}}_{+2}\left(z, 1^{-1+1}, 2^{+1-1}\right) \times M_{n-1}\left(p^{-2}, \cdots, n\right)\Big] \\ &= \varepsilon^{abcd}\varepsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots, n\right)\right. \\ &\quad \left. + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots, n\right)\right] \end{aligned} \tag{B.32}$$

$$\begin{aligned} M_n\left(1^{abcu;0}, 2^{drst;0}, \cdots, n\right) &= \varepsilon^{abce}\varepsilon^{rstv} M_n\left(1^{(e,-\frac{1}{2})\otimes(u,\frac{1}{2})}, 2^{(d,+\frac{1}{2})\otimes(v,-\frac{1}{2})}, \cdots, n\right) \\ &= \varepsilon^{abce}\varepsilon^{rstv}\delta^d_e\delta^u_v\Big[\mathrm{Split}^{\mathrm{SG}}_{-2}\left(z, 1^{-\frac{1}{2}+\frac{1}{2}}, 2^{\frac{1}{2}-\frac{1}{2}}\right) \times M_{n-1}\left(p^{+2}, \cdots, n\right) \\ &\quad + \mathrm{Split}^{\mathrm{SG}}_{+2}\left(z, 1^{-\frac{1}{2}+\frac{1}{2}}, 2^{+\frac{1}{2}-\frac{1}{2}}\right) \times M_{n-1}\left(p^{-2}, \cdots, n\right)\Big] \\ &= \varepsilon^{abcd}\varepsilon^{rstu}\left[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{+2}, \cdots, n\right)\right. \\ &\quad \left. + \frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{-2}, \cdots, n\right)\right] \end{aligned} \tag{B.33}$$

Similarly,

$$M_n\left(1^{arst;0},2^{bcdu;0},\cdots,n\right)=\varepsilon^{rstu}\varepsilon^{abcd}\Big[\frac{\omega_2\omega_1}{\omega_p^2}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{+2},\cdots,n\right)$$
$$+\frac{\omega_1\omega_2}{\omega_p^2}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-2},\cdots,n\right)\Big]$$

B.2.2 For Unlike Spins

We now use the splits for mixed helicities listed in Appendix B.1 and the factorisation of R-symmetry indices mentioned in the calculation of collinear limit for like spins.

Graviton-Gravitino
We have

$$\begin{aligned}M_n\left(1^{+2},2^{r;+\frac{3}{2}},\cdots,n\right)&=M_n\left(1^{(1\otimes 1)},2^{1\otimes(r;\frac{1}{2})},\cdots,n\right)\\&=\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{1+1},2^{1+\frac{1}{2}}\right)\times M_{n-1}\left(p^{r;+\frac{3}{2}},\cdots,n\right)\\&=\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{r;+\frac{3}{2}},\cdots,n\right)\end{aligned}\tag{B.34}$$

$$\begin{aligned}M_n\left(1^{+2},2_r^{-\frac{3}{2}},\cdots,n\right)&=M_n\left(1^{(1\otimes 1)},2^{-1\otimes(r;-\frac{1}{2})},\cdots,n\right)\\&=\mathrm{Split}^{\mathrm{SG}}_{+\frac{3}{2}}\left(z,1^{1+1},2^{-1-\frac{1}{2}}\right)\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)\\&=\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)\end{aligned}\tag{B.35}$$

Similarly we have

$$M_n\left(1^{+2},2^{a;+\frac{3}{2}},\cdots,n\right)=\frac{\omega_p^{\frac{3}{2}}}{\omega_2^{\frac{1}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{a;+\frac{3}{2}},\cdots,n\right)$$

$$M_n\left(1^{+2},2_a^{-\frac{3}{2}},\cdots,n\right)=\frac{\omega_2^{\frac{5}{2}}}{\omega_p^{\frac{3}{2}}\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_a^{-\frac{3}{2}},\cdots,n\right)$$

Other helicity combination of graviton and gravitino can be obtained by flipping the indices along with $z_{12}\leftrightarrow\bar{z}_{12}$.

Graviton-Graviphoton

$$\begin{aligned}M_n\left(1^{+2},2^{ab;+1},\cdots,n\right)&=M_n\left(1^{(1\otimes 1)},2^{(ab;0)\otimes 1},\cdots,n\right)\\&=\mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{1+1},2^{0+1}\right)\times M_{n-1}\left(p^{ab;+1},\cdots,n\right)\\&=\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ab;+1},\cdots,n\right)\end{aligned}\tag{B.36}$$

$$\begin{aligned} M_n\left(1^{+2},2^{rs;+1},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{1\otimes(rs;0)},\cdots,n\right) \\ &= \mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{1+1},2^{1+0}\right)\times M_{n-1}\left(p^{rs;+1},\cdots,n\right) \\ &= \frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{rs;+1},\cdots,n\right) \end{aligned} \tag{B.37}$$

$$\begin{aligned} M_n\left(1^{+2},2^{ar;+1},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{(a;\frac{1}{2})\otimes(r;\frac{1}{2})},\cdots,n\right) \\ &= \mathrm{Split}^{\mathrm{SG}}_{-1}\left(z,1^{1+1},2^{\frac{1}{2}+\frac{1}{2}}\right)\times M_{n-1}\left(p^{ar;+1},\cdots,n\right) \\ &= \frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ar;+1},\cdots,n\right) \end{aligned} \tag{B.38}$$

$$\begin{aligned} M_n\left(1^{+2},2^{-1}_{ab},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{(ab;0)\otimes -1},\cdots,n\right) \\ &= \mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+1},2^{0-1}\right)\times M_{n-1}\left(p^{-1}_{ab},\cdots,n\right) \\ &= \frac{\omega_2^2}{\omega_p\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{ab},\cdots,n\right) \end{aligned} \tag{B.39}$$

$$\begin{aligned} M_n\left(1^{+2},2^{-1}_{rs},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{-1\otimes(rs;0)},\cdots,n\right) \\ &= \mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+1},2^{-1+0}\right)\times M_{n-1}\left(p^{-1}_{rs},\cdots,n\right) \\ &= \frac{\omega_2^2}{\omega_p\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{rs},\cdots,n\right) \end{aligned} \tag{B.40}$$

$$\begin{aligned} M_n\left(1^{+2},2^{-1}_{ar},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{(a;-\frac{1}{2})\otimes(r;-\frac{1}{2})},\cdots,n\right) \\ &= \mathrm{Split}^{\mathrm{SG}}_{+1}\left(z,1^{1+1},2^{-\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}\left(p^{-1}_{ar},\cdots,n\right) \\ &= \frac{\omega_2^2}{\omega_1\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{ar},\cdots,n\right) \end{aligned} \tag{B.41}$$

Similarly we can calculate the collinear helicity combinations of Graviphotons with negative helicity Gravitons.
Hence

$$M_n\left(1^{+2},2^{-1}_{AB},\cdots,n\right) = \frac{\omega_2^2}{\omega_1\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{AB},\cdots,n\right) \tag{B.42}$$

Graviton-Graviphotino

$$\begin{aligned} M_n\left(1^{+2},2^{abr;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{(ab;0)\otimes(r;\frac{1}{2})},\cdots,n\right) \\ &= \mathrm{Split}^{\mathrm{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{0+\frac{1}{2}}\right)\times M_{n-1}\left(p^{abr;+\frac{1}{2}},\cdots,n\right) \\ &= \frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abr;+\frac{1}{2}},\cdots,n\right) \end{aligned} \tag{B.43}$$

Similarly,

$$M_n\left(1^{+2},2^{ars;+\frac{1}{2}},\cdots,n\right) = \frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{ars;+\frac{1}{2}},\cdots,n\right)$$

$$\begin{aligned}M_n\left(1^{+2},2^{abc;+\frac{1}{2}},\cdots,n\right) &= -\varepsilon^{abcd}M_n\left(1^{(1\otimes 1)},2^{(d;-\frac{1}{2})\otimes 1},\cdots,n\right)\\ &= -\frac{1}{3!}\varepsilon^{abcd}\varepsilon_{defg}\text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z,1^{1+1},2^{-\frac{1}{2}+1}\right)\times M_{n-1}\left(p^{efg;+\frac{1}{2}},\cdots,n\right)\\ &= \frac{1}{3!}\delta^{abc}_{efg}\frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{efg;+\frac{1}{2}},\cdots,n\right)\\ &= \frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{abc;+\frac{1}{2}},\cdots,n\right)\end{aligned} \tag{B.44}$$

Here we are using change of basis as a redefinition for the fields in SYM:

$$\Gamma^-_a \equiv \frac{1}{3!}\varepsilon_{abcd}\Gamma^{-bcd}$$

Similarly,

$$M_n\left(1^{+2},2^{rst;+\frac{1}{2}},\cdots,n\right) = \frac{\sqrt{\omega_2\omega_p}}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{rst;+\frac{1}{2}},\cdots,n\right)$$

$$\begin{aligned}M_n\left(1^{+2},2^{-\frac{1}{2}}_{abr},\cdots,n\right) &= M_n\left(1^{(1\otimes 1)},2^{(ab;0)\otimes(r;-\frac{1}{2})},\cdots,n\right)\\ &= \text{Split}^{\text{SG}}_{+\frac{1}{2}}\left(z,1^{1+1},2^{0-\frac{1}{2}}\right)\times M_{n-1}\left(p^{-\frac{1}{2}}_{abr},\cdots,n\right)\\ &= \frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{abr},\cdots,n\right)\end{aligned} \tag{B.45}$$

Similarly,

$$M_n\left(1^{+2},2^{-\frac{1}{2}}_{ars},\cdots,n\right) = \frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{ars},\cdots,n\right)$$

$$\begin{aligned}M_n\left(1^{+2},2^{-\frac{1}{2}}_{abc},\cdots,n\right) &= \varepsilon_{abcd}M_n\left(1^{(1\otimes 1)},2^{(d;\frac{1}{2})\otimes -1},\cdots,n\right)\\ &= \frac{1}{3!}\varepsilon_{abcd}\varepsilon^{defg}\text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{1+1},2^{\frac{1}{2}-1}\right)\times M_{n-1}\left(p^{-\frac{1}{2}}_{efg;},\cdots,n\right)\\ &= -\frac{1}{3!}\delta^{efg}_{abc}\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{efg;},\cdots,n\right)\\ &= -\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{abc},\cdots,n\right)\end{aligned} \tag{B.46}$$

Similarly

$$M_n\left(1^{+2},2^{-\frac{1}{2}}_{rst},\cdots,n\right) = -\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{1}{2}}\,\omega_1}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{rst},\cdots,n\right).$$

Graviton-Scalar

Since the split factors corresponding to all factorisations of the R-symmetry indices is the same, the collinear limit can be uniformly written as

$$M_n\left(1^{+2}, 2^{ABCD;0}, \cdots, n\right) = \frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p^{ABCD;0}, \cdots, n\right) \tag{B.47}$$

Gravitino-Graviphoton

$$\begin{aligned} M_n\left(1^{a;+\frac{3}{2}}, 2^{bc;+1}, \cdots, n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes 1}, 2^{(bc;0)\otimes 1}, \cdots, n\right) \\ &= \text{Split}^{\text{SG}}_{-\frac{1}{2}}\left(z, 1^{\frac{1}{2}+1}, 2^{0+1}\right) \times M_{n-1}\left(p^{abc;\frac{1}{2}}, \cdots, n\right) \\ &= \sqrt{\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}\left(p^{abc;+\frac{1}{2}}, \cdots, n\right) \end{aligned} \tag{B.48}$$

Similarly,

$$M_n\left(1^{r;+\frac{3}{2}}, 2^{st;+1}, \cdots, n\right) = \sqrt{\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}\left(p^{rst;+\frac{1}{2}}, \cdots, n\right)$$

$$M_n\left(1^{a;+\frac{3}{2}}, 2^{rs;+1}, \cdots, n\right) = \sqrt{\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}\left(p^{ars;+\frac{1}{2}}, \cdots, n\right)$$

$$M_n\left(1^{r;+\frac{3}{2}}, 2^{ab;+1}, \cdots, n\right) = \sqrt{\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}\left(p^{rab;+\frac{1}{2}}, \cdots, n\right)$$

In conclusion, we can write

$$M_n\left(1^{A;+\frac{3}{2}}, 2^{BC;+1}, \cdots, n\right) = \sqrt{\frac{\omega_p}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}} \times M_{n-1}\left(p^{ABC;+\frac{1}{2}}, \cdots, n\right) \tag{B.49}$$

$$\begin{aligned} M_n\left(1^{a;+\frac{3}{2}}, 2^{-1}_{bc}, \cdots, n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes 1}, 2^{(bc;0)\otimes 1}, \cdots, n\right) \\ &= \frac{1}{2!}\varepsilon_{bcde}\varepsilon^{adef}\ \text{Split}^{\text{SG}}_{\frac{3}{2}}\left(z, 1^{\frac{1}{2}+1}, 2^{0-1}\right) \times M_{n-1}\left(p_f^{-\frac{3}{2}}, \cdots, n\right) \\ &= \delta^{af}_{bc}\frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_f^{-\frac{3}{2}}, \cdots, n\right) \\ &= \frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} 2!\delta^a_{[b} \times M_{n-1}\left(p_{c]}^{-\frac{3}{2}}, \cdots, n\right) \end{aligned} \tag{B.50}$$

$$M_n\left(1^{r;+\frac{3}{2}}, 2^{-1}_{st}, \cdots, n\right) = \frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} 2!\delta^r_{[s} \times M_{n-1}\left(p_{t]}^{-\frac{3}{2}}, \cdots, n\right).$$

Hence for any $1 \leq A, B \leq 8$ we have,

$$M_n\left(1^{A;+\frac{3}{2}}, 2^{-1}_{BC}, \cdots, n\right) = \frac{\omega_2^2}{\omega_p^{\frac{3}{2}}\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}} 2!\delta^A_{[B} \times M_{n-1}\left(p_{C]}^{-\frac{3}{2}}, \cdots, n\right). \tag{B.51}$$

Gravitino-Graviphotino

$$\begin{aligned} M_n\left(1^{a;+\frac{3}{2}},2^{brs;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes 1},2^{(b;\frac{1}{2})\otimes(rs;0)},\cdots,n\right) \\ &= \mathrm{Split}_0^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{\frac{1}{2}+0}\right)\times M_{n-1}\left(p^{abrs;0},\cdots,n\right) \\ &= \sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{abrs;0},\cdots,n\right) \end{aligned} \tag{B.52}$$

Similarly for all other factorisations the split factors will remain the same for two same helicity Gravitino and Graviphotino pair,

$$M_n\left(1^{a;+\frac{3}{2}},2^{bcr;+\frac{1}{2}},\cdots,n\right) = \sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{abcr;0},\cdots,n\right)$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{sta;+\frac{1}{2}},\cdots,n\right) = \sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{rsta;0},\cdots,n\right)$$

$$M_n\left(1^{r;+\frac{3}{2}},2^{abs;+\frac{1}{2}},\cdots,n\right) = \sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{abrs;0},\cdots,n\right)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{bcd;+\frac{1}{2}},\cdots,n\right) = \sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{abcd;0},\cdots,n\right)$$

$$M_n\left(1^{a;+\frac{3}{2}},2^{rst;+\frac{1}{2}},\cdots,n\right) = \sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{arst;0},\cdots,n\right)$$

Collecting all of them, we can write

$$M_n\left(1^{A;+\frac{3}{2}},2^{BCD;+\frac{1}{2}},\cdots,n\right) = \sqrt{\frac{\omega_2}{\omega_1}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p^{ABCD;0},\cdots,n\right) \tag{B.53}$$

$$\begin{aligned} M_n\left(1^{a;+\frac{3}{2}},2^{-\frac{1}{2}}_{bcr},\cdots,n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes 1},2^{(bc;0)\otimes(r;-\frac{1}{2})},\cdots,n\right) \\ &= -\frac{1}{2!}\varepsilon_{bcde}\varepsilon^{deaf}\ \mathrm{Split}_{+1}^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+1},2^{0-\frac{1}{2}}\right)\times M_{n-1}\left(p^{-1}_{fr},\cdots,n\right) \\ &= -\delta^{af}_{bc}\ \frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-1}_{fr},\cdots,n\right) \\ &= -2!\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta^a_{[b}\times M_{n-1}\left(p^{-1}_{c]r},\cdots,n\right) \\ &= -3!\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta^a_{[b}\times M_{n-1}\left(p^{-1}_{cr]},\cdots,n\right) \end{aligned} \tag{B.54}$$

where we used the fact that $\delta^a_r = 0$ to write

$$\begin{aligned} 2!\delta^a_{[b}p^{-1}_{c]r} &= \delta^a_b p^{-1}_{cr} - \delta^a_b p^{-1}_{rc} + \delta^a_c p^{-1}_{rb} - \delta^a_c p^{-1}_{br} + \delta^a_r p^{-1}_{bc} - \delta^a_r p^{-1}_{cb} \\ &= 3!\delta^a_{[b}p^{-1}_{cr]}. \end{aligned} \tag{B.55}$$

$$\begin{aligned}
M_n\left(1^{a;+\frac{3}{2}}, 2_{brs}^{-\frac{1}{2}}, \cdots, n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes 1}, 2^{(b;-\frac{1}{2})\otimes(rs;0)}, \cdots, n\right) \\
&= \delta_b^a \, \text{Split}_{+1}^{\text{SG}}\left(z, 1^{\frac{1}{2}+1}, 2^{-\frac{1}{2}+0}\right) \times M_{n-1}\left(p_{rs}^{-1}, \cdots, n\right) \\
&= \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \delta_b^a \times M_{n-1}\left(p_{rs}^{-1}, \cdots, n\right).
\end{aligned} \tag{B.56}$$

Similarly we have

$$\begin{aligned}
M_n\left(1^{r;+\frac{3}{2}}, 2_{ast}^{-\frac{1}{2}}, \cdots, n\right) &= -\frac{1}{2!}\varepsilon_{stuv}\varepsilon^{uvrw} \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{wa}^{-1}, \cdots, n\right) \\
&= -3! \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \delta_{[s}^r \times M_{n-1}\left(p_{ta]}^{-1}, \cdots, n\right)
\end{aligned}$$

$$M_n\left(1^{t;+\frac{3}{2}}, 2_{rab}^{-\frac{1}{2}}, \cdots, n\right) = \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \delta_r^t \times M_{n-1}\left(p_{ab}^{-1}, \cdots, n\right)$$

$$\begin{aligned}
M_n\left(1^{a;+\frac{3}{2}}, 2_{bcd}^{-\frac{1}{2}}, \cdots, n\right) &= \varepsilon_{bcde} M_n\left(1^{(a;\frac{1}{2})\otimes 1}, 2^{(e;+\frac{1}{2})\otimes -1}, \cdots, n\right) \\
&= -\frac{1}{2!}\varepsilon_{bcde}\varepsilon^{aefg} \, \text{Split}_{+1}^{\text{SG}}\left(z, 1^{\frac{1}{2}+1}, 2^{\frac{1}{2}-1}\right) \times M_{n-1}\left(p_{fg}^{-1}, \cdots, n\right) \\
&= -\frac{1}{2!}\varepsilon_{bcde}\varepsilon^{afge} \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{fg}^{-1}, \cdots, n\right) \\
&= -\frac{1}{2!}\delta_{bcd}^{afg} \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \times M_{n-1}\left(p_{fg}^{-1}, \cdots, n\right) \\
&= 3 \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \delta_{[b}^a \times M_{n-1}\left(p_{cd]}^{-1}, \cdots, n\right)
\end{aligned} \tag{B.57}$$

Note that the second ε^{aefg} comes from the fact that we are lowering the index of he scalar in $\mathcal{N}=4$ SYM in the factorisation of the negative helicity gluon. The factorisation looks as

$$G_{fg}^{-1} = \Phi_{fg} \otimes G^{-1} = -\frac{1}{2!}\varepsilon_{aefg}\Phi^{ae} \otimes G^{-1} =: -\frac{1}{2!}\varepsilon_{aefg} G^{ae-1}, \tag{B.58}$$

where G_{fg} is the gluon in $\mathcal{N}=8$ supergravity and G is the gluon in $\mathcal{N}=4$ SYM.

$$M_n\left(1^{d;+\frac{3}{2}}, 2_{abc}^{-\frac{1}{2}}, \cdots, n\right) = -3 \frac{\omega_2^{\frac{3}{2}}}{\omega_p \omega_1^{\frac{1}{2}}} \frac{\bar{z}_{12}}{z_{12}} \delta_{[a}^d \times M_{n-1}\left(p_{bc]}^{-1}, \cdots, n\right)$$

Thus we have the collinear limit

$$M_n\left(1^{u;+\frac{3}{2}},2^{-\frac{1}{2}}_{rst},\cdots,n\right)=-3\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\delta^u_{[r}\times M_{n-1}\left(p^{-1}_{st]},\cdots,n\right)$$

Hence

$$\begin{aligned}M_n\left(1^{A;+\frac{3}{2}},2^{-\frac{1}{2}}_{BCD},\cdots,n\right)=-\frac{\omega_2^{\frac{3}{2}}}{\omega_p\omega_1^{\frac{1}{2}}}\frac{\bar{z}_{12}}{z_{12}}\Big[\delta^A_B M_{n-1}\left(p^{-1}_{CD},\cdots,n\right)\\+\delta^A_C M_{n-1}\left(p^{-1}_{DB},\cdots,n\right)+\delta^A_D M_{n-1}\left(p^{-1}_{BC},\cdots,n\right)\Big]\end{aligned}\tag{B.59}$$

Similarly we get the same splitting factors for all other factorisation channels.

Gravitino-Scalar

$$\begin{aligned}M_n\left(1^{a;+\frac{3}{2}},2^{bcrs;0},\cdots,n\right)&=M_n\left(1^{(a;\frac{1}{2})\otimes 1},2^{(bc;0)\otimes(rs;0)},\cdots,n\right)\\&=\varepsilon^{rstu}\varepsilon^{abcd}\text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{0+0}\right)\times M_{n-1}\left(p^{-\frac{1}{2}}_{dtu},\cdots,n\right)\\&=\varepsilon^{rstu}\varepsilon^{abcd}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{drs},\cdots,n\right)\end{aligned}\tag{B.60}$$

$$\begin{aligned}M_n\left(1^{a;+\frac{3}{2}},2^{rstu;0},\cdots,n\right)&=M_n\left(1^{(a;\frac{1}{2})\otimes 1},2^{rstu;(1\otimes 1)},\cdots,n\right)\\&=-\varepsilon^{rstu}\varepsilon^{abcd}\text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{1-1}\right)\times M_{n-1}\left(p^{-\frac{1}{2}}_{bcd},\cdots,n\right)\\&=-\varepsilon^{abcd}\varepsilon^{rstu}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{bcd},\cdots,n\right)\end{aligned}\tag{B.61}$$

where we lowered the index on gluino in the SYM theory.

$$\begin{aligned}M_n\left(1^{r;+\frac{3}{2}},2^{abcd;0},\cdots,n\right)&=M_n\left(1^{(r;\frac{1}{2})\otimes 1},2^{abcd;(1\otimes 1)},\cdots,n\right)\\&=-\varepsilon^{abcd}\varepsilon^{rstu}\text{Split}^{\text{SG}}_{\frac{1}{2}}\left(z,1^{\frac{1}{2}+1},2^{-1+1}\right)\times M_{n-1}\left(p^{-\frac{1}{2}}_{stu},\cdots,n\right)\\&=-\varepsilon^{abcd}\varepsilon^{rstu}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{stu},\cdots,n\right)\end{aligned}\tag{B.62}$$

Hence we can write the above in simplified form as

$$M_n\left(1^{A;+\frac{3}{2}},2^{BCDE;0},\cdots,n\right)=-\frac{1}{3!}\varepsilon^{ABCDEFGH}\frac{\omega_2}{\sqrt{\omega_1\omega_p}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p^{-\frac{1}{2}}_{FGH},\cdots,n\right)\tag{B.63}$$

Similarly we can have the relations for opposite helicity collinear pair.

Graviphoton-Graviphotino

$$
\begin{aligned}
M_n\left(1^{ab;+1},2^{cdr;+\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{(ab;0)\otimes 1},2^{(cd;0)\otimes(r;\frac{1}{2})},\cdots,n\right)\\
&= \frac{1}{3!}\varepsilon^{abcd}\varepsilon^{rstu}\,\mathrm{Split}^{\mathrm{SG}}_{\frac{1}{2}}\left(z,1^{0+1},2^{0+\frac{1}{2}}\right)\times M_{n-1}\left(p_{stu}^{-\frac{1}{2}},\cdots,n\right)\\
&= \frac{1}{3!}\varepsilon^{abcd}\varepsilon^{rstu}\,\sqrt{\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}}\times M_{n-1}\left(p_{stu}^{-\frac{1}{2}},\cdots,n\right)
\end{aligned} \tag{B.64}
$$

This is true for all other factorisation channels of both positive helicity Graviphoton and Graviphotino collinear pair. Similarly we can have the amplitude for negative helicity collinear pairs.

$$
\begin{aligned}
M_n\left(1^{ab;+1},2_{cdr;-\frac{1}{2}},\cdots,n\right) &= M_n\left(1^{(ab;0)\otimes 1},2^{(cd;0)\otimes(r;\frac{1}{2})},\cdots,n\right)\\
&= -\frac{1}{2!}\varepsilon^{abef}\varepsilon_{cdef}\,\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+1},2^{0-\frac{1}{2}}\right)\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)\\
&= -\delta^{ab}_{cd}\,\frac{\omega_2^{\frac{3}{2}}}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_r^{-\frac{3}{2}},\cdots,n\right)
\end{aligned} \tag{B.65}
$$

All other factorisation channels also correspond to the same collinear divergence factor and we get the other amplitudes in the usual way by flipping the helicity and $z_{12}\leftrightarrow\bar{z}_{12}$.

Graviphoton-Scalar

$$
\begin{aligned}
M_n\left(1^{ab;+1},2^{cdrs;0},\cdots,n\right) &= M_n\left(1^{(ab;0)\otimes 1},2^{(cd;0)\otimes(rs;0)},\cdots,n\right)\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{0+1},2^{0+0}\right)\times M_{n-1}\left(p_{tu}^{-1},\cdots,n\right) \qquad (\mathrm{B.66})\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{tu}^{-1},\cdots,n\right)
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{rs;+1},2^{abtu;0},\cdots,n\right) &= M_n\left(1^{1\otimes(rs;0)},2^{(ab;0)\otimes(tu;0)},\cdots,n\right)\\
&= \varepsilon^{rstu}\varepsilon^{abcd}\,\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{1+0},2^{0+0}\right)\times M_{n-1}\left(p_{cd}^{-1},\cdots,n\right) \qquad (\mathrm{B.67})\\
&= \varepsilon^{rstu}\varepsilon^{abcd}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{cd}^{-1},\cdots,n\right)
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{ar;+1},2^{bcst;0},\cdots,n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes(r;\frac{1}{2})},2^{(bc;0)\otimes(st;0)},\cdots,n\right)\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p_{du}^{-1},\cdots,n\right) \qquad (\mathrm{B.68})\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{du}^{-1},\cdots,n\right)
\end{aligned}
$$

$$
\begin{aligned}
M_n\left(1^{ab;+1},2^{cdef;0},\cdots,n\right) &= -M_n\left(1^{(ab;0)\otimes 1},2^{cdef;(-1\otimes 1)},\cdots,n\right)\\
&= -\varepsilon^{cdef}\varepsilon^{abgh}\,\mathrm{Split}^{\mathrm{SG}}_{1}\left(z,1^{0+1},2^{-1+1}\right)\times M_{n-1}\left(p_{gh}^{-1},\cdots,n\right)\\
&= -\varepsilon^{cdef}\varepsilon^{abgh}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{gh}^{-1},\cdots,n\right)
\end{aligned} \tag{B.69}
$$

$$\begin{aligned} M_n\left(1^{rs;+1},2^{cdef;0},\cdots,n\right) &= -M_n\left(1^{1\otimes(rs;0)},2^{cdef;(-1\otimes 1)},\cdots,n\right) \\ &= -\varepsilon^{cdef}\varepsilon^{rstu}\,\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{1+0},2^{-1+1}\right)\times M_{n-1}\left(p_{tu}^{-1},\cdots,n\right) \\ &= -\varepsilon^{cdef}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{tu}^{-1},\cdots,n\right) \end{aligned} \tag{B.70}$$

$$\begin{aligned} M_n\left(1^{ar;+1},2^{bcds;0},\cdots,n\right) &= -M_n\left(1^{(a;\frac{1}{2})\otimes(r;\frac{1}{2})},2^{bcds;(-\frac{1}{2}+\frac{1}{2})},\cdots,n\right) \\ &= -\varepsilon^{abcd}\varepsilon^{rstu}\,\mathrm{Split}_1^{\mathrm{SG}}\left(z,1^{\frac{1}{2}+\frac{1}{2}},2^{-\frac{1}{2}+\frac{1}{2}}\right)\times M_{n-1}\left(p_{tu}^{-1},\cdots,n\right) \\ &= -\varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_2}{\omega_p}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_{tu}^{-1},\cdots,n\right) \end{aligned} \tag{B.71}$$

Similarly we can write for other remaining factorisation channels.

$$\begin{aligned} M_n\left(1_{ab}^{-1},2^{cdef;0},\cdots,n\right) &= -M_n\left(1^{(ab;0)\otimes -1},2^{cdef;(-1\otimes 1)},\cdots,n\right) \\ &= -\varepsilon^{cdef}\varepsilon_{abgh}\,\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{0-1},2^{-1+1}\right)\times M_{n-1}\left(p^{gh;+1},\cdots,n\right) \\ &= -\varepsilon^{cdef}\varepsilon_{abgh}\,\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{gh;+1},\cdots,n\right) \end{aligned} \tag{B.72}$$

$$\begin{aligned} M_n\left(1_{ar}^{-1},2^{bcds;0},\cdots,n\right) &= -M_n\left(1^{(a;-\frac{1}{2})\otimes(r;-\frac{1}{2})},2^{bcds;(-\frac{1}{2}\otimes\frac{1}{2})},\cdots,n\right) \\ &= -\varepsilon^{bcde}\delta_r^s\,\varepsilon_{aefg}\,\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{-\frac{1}{2}+\frac{1}{2}}\right)\times M_{n-1}\left(p^{fg;+1},\cdots,n\right) \\ &= -\varepsilon^{bcde}\delta_r^s\varepsilon_{aefg}\,\frac{\omega_1}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{fg;+1},\cdots,n\right) \end{aligned} \tag{B.73}$$

$$\begin{aligned} M_n\left(1_{ar}^{-1},2^{bcst;0},\cdots,n\right) &= -M_n\left(1^{(a;\frac{1}{2})\otimes(r;\frac{1}{2})},2^{(bc;0)\otimes(st;0)},\cdots,n\right) \\ &= \Big[-\delta_a^b\delta_r^s\,\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p^{ct;+1},\cdots,n\right) \\ &+\delta_a^c\delta_r^t\,\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p^{bs;+1},\cdots,n\right) \\ &+\delta_a^b\delta_r^t\,\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p^{cs;+1},\cdots,n\right) \\ &-\delta_a^c\delta_r^s\,\mathrm{Split}_{-1}^{\mathrm{SG}}\left(z,1^{-\frac{1}{2}-\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p^{bt;+1},\cdots,n\right)\Big] \\ &= -\frac{\omega_2}{\omega_p}\frac{z_{12}}{\bar{z}_{12}}4!\delta_a^{[b}\delta_r^s\,M_{n-1}\left(p^{tc];+1},\cdots,n\right) \end{aligned} \tag{B.74}$$

Note that the above expression contains 16 terms but only four terms are nonzero since $\delta_r^a = 0$.

Graviphotino-Scalar

$$\begin{aligned}
M_n\left(1^{abr;+\frac{1}{2}},2^{cdst;0},\cdots,n\right) &= M_n\left(1^{(ab;0)\otimes(r;\frac{1}{2})},2^{(cd;0)\otimes(st;0)},\cdots,n\right)\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+\frac{1}{2}},2^{0+0}\right)\times M_{n-1}\left(p_u^{-\frac{3}{2}},\cdots,n\right)\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_u^{-\frac{3}{2}},\cdots,n\right)
\end{aligned} \tag{B.75}$$

$$\begin{aligned}
M_n\left(1^{abr;+\frac{1}{2}},2^{cstu;0},\cdots,n\right) &= -M_n\left(1^{(ab;0)\otimes(r;\frac{1}{2})},2^{cstu;(\frac{1}{2}\otimes-\frac{1}{2})},\cdots,n\right)\\
&= -\varepsilon^{abcd}\varepsilon^{rstu}\,\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{0+\frac{1}{2}},2^{\frac{1}{2}-\frac{1}{2}}\right)\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)\\
&= -\varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)
\end{aligned} \tag{B.76}$$

$$\begin{aligned}
M_n\left(1^{ars;+\frac{1}{2}},2^{bctu;0},\cdots,n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes(rs;0)},2^{(bc;0)\otimes(tu;0)},\cdots,n\right)\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\mathrm{Split}^{\mathrm{SG}}_{\frac{3}{2}}\left(z,1^{\frac{1}{2}+0},2^{0+0}\right)\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)\\
&= \varepsilon^{abcd}\varepsilon^{rstu}\,\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{\bar{z}_{12}}{z_{12}}\times M_{n-1}\left(p_d^{-\frac{3}{2}},\cdots,n\right)
\end{aligned} \tag{B.77}$$

$$\begin{aligned}
M_n\left(1^{-\frac{1}{2}}_{ars},2^{bctu;0},\cdots,n\right) &= M_n\left(1^{(a;\frac{1}{2})\otimes(rs;0)},2^{(bc;0)\otimes(tu;0)},\cdots,n\right)\\
&= -\frac{1}{2!}\varepsilon^{tuvw}\varepsilon_{vwrs}2!\delta_a^{[b}\,\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-\frac{1}{2}+0},2^{0+0}\right)\times M_{n-1}\left(p^{c];+\frac{3}{2}},\cdots,n\right)\\
&= -2!\delta_{rs}^{tu}\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}}\delta_a^{[b}\times M_{n-1}\left(p^{c];+\frac{3}{2}},\cdots,n\right)
\end{aligned} \tag{B.78}$$

$$\begin{aligned}
M_n\left(1^{-\frac{1}{2}}_{ars},2^{btuv\;0},\cdots,n\right) &= -M_n\left(1^{(a;\frac{1}{2})\otimes(rs;0)},2^{btuv;(\frac{1}{2}\otimes-\frac{1}{2})},\cdots,n\right)\\
&= -\delta_a^b\varepsilon^{tuvw}\varepsilon_{wrsx}\,\mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z,1^{-\frac{1}{2}+0},2^{-\frac{1}{2}+\frac{1}{2}}\right)\times M_{n-1}\left(p^{x;+\frac{3}{2}},\cdots,n\right)\\
&= -\delta_a^b\varepsilon^{tuvw}\varepsilon_{wrsx}\,\frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}}\frac{z_{12}}{\bar{z}_{12}}\times M_{n-1}\left(p^{x;+\frac{3}{2}},\cdots,n\right)
\end{aligned} \tag{B.79}$$

$$
\begin{aligned}
M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{avwx;0}, \cdots, n\right) &= -M_n\left(1^{rst;(-1\otimes\frac{1}{2})}, 2^{avwx;(\frac{1}{2}\otimes-\frac{1}{2})}, \cdots, n\right) \\
&= -\varepsilon_{rstu}\varepsilon^{vwxy}\delta_y^u \, \mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z, 1^{-1+\frac{1}{2}}, 2^{\frac{1}{2}-\frac{1}{2}}\right) \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right) \\
&= -\varepsilon_{rstu}\varepsilon^{vwxu} \, \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}} \frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{a;+\frac{3}{2}}, \cdots, n\right)
\end{aligned}
\tag{B.80}
$$

$$
\begin{aligned}
M_n\left(1_{rst}^{-\frac{1}{2}}, 2^{uvwx;0}, \cdots, n\right) &= -M_n\left(1^{rst;(-1\otimes\frac{1}{2})}, 2^{uvwx;(1\otimes-1)}, \cdots, n\right) \\
&= -\varepsilon_{rsty}\varepsilon^{uvwx} \, \mathrm{Split}^{\mathrm{SG}}_{-\frac{3}{2}}\left(z, 1^{-1+\frac{1}{2}}, 2^{1-1}\right) \times M_{n-1}\left(p^{y;+\frac{3}{2}}, \cdots, n\right) \\
&= -\varepsilon_{rsty}\varepsilon^{uvwx} \, \frac{\omega_1^{\frac{1}{2}}\omega_2}{\omega_p^{\frac{3}{2}}} \frac{z_{12}}{\bar{z}_{12}} \times M_{n-1}\left(p^{y;+\frac{3}{2}}, \cdots, n\right)
\end{aligned}
\tag{B.81}
$$

APPENDIX C

OPES OF COMPONENT FIELDS IN SUGRA

C.1 OPEs of like and unlike spins

Using (5.7), we can extract the rest of the OPEs from the collinear singularities of the amplitudes calculated in [392]. In the following, the zero, one, two, three, and four index operators are, respectively, graviton, gravitino, graviphoton, gravitino, and scalar operators.

C.1.1 Same spin OPEs

$$\begin{aligned}
\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{B}_{\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2) &\sim \frac{\bar{z}_{12}}{z_{12}} B\left(\Delta_1-\frac{1}{2},\Delta_2-\frac{1}{2}\right)\mathcal{O}^{AB}_{\Delta_1+\Delta_2,+1}(z_2,\bar{z}_2)\\
\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}_{B\,\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2) &\sim \frac{\bar{z}_{12}}{z_{12}}\,\delta^A_B\, B\left(\Delta_1-\frac{1}{2},\Delta_2+\frac{5}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2)\\
&\quad+\frac{z_{12}}{\bar{z}_{12}}\,\delta^A_B\, B\left(\Delta_1+\frac{5}{2},\Delta_2-\frac{1}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2)
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
\mathcal{O}^{AB}_{\Delta_1,+1}(z_1,\bar{z}_1)\,\mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2) &\sim \frac{\bar{z}_{12}}{z_{12}} B(\Delta_1,\Delta_2)\,\mathcal{O}^{ABCD}_{\Delta_1+\Delta_2,0}(z_2,\bar{z}_2)\\
\mathcal{O}^{AB}_{\Delta_1,+1}(z_1,\bar{z}_1)\,\mathcal{O}_{CD;\,\Delta_2,-1}(z_2,\bar{z}_2) &\sim -\delta^{AB}_{CD}\Big[\frac{\bar{z}_{12}}{z_{12}} B(\Delta_1,\Delta_2+2)\,\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2)\\
&\quad+\frac{z_{12}}{\bar{z}_{12}} B(\Delta_1+2,\Delta_2)\,\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2)\Big]
\end{aligned} \tag{C.2}$$

In the following, the notation is $a,b,c,\dots \in \{1,2,3,4\}$ and $r,s,t,\dots \in \{5,6,7,8\}$. See [392] for details.

$$\begin{aligned}
\mathcal{O}^{ars}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{btu}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) &\sim \varepsilon^{rstu}\varepsilon^{abcd}\frac{z_{12}}{\bar{z}_{12}} B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{cd;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2)\\
\mathcal{O}^{ars}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{bct}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) &\sim \varepsilon^{rstu}\varepsilon^{abcd}\frac{z_{12}}{\bar{z}_{12}} B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{ud;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2)\\
\mathcal{O}^{rst}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{abc}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2) &\sim \varepsilon^{rstu}\varepsilon^{abcd}\frac{z_{12}}{\bar{z}_{12}} B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{ud;\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2)\\
\mathcal{O}^{ars}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\,\mathcal{O}_{btu;\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2) &\sim \varepsilon_{tuvw}\varepsilon^{rsvw}\delta^a_b\Big[\frac{z_{12}}{\bar{z}_{12}} B\left(\Delta_1+\frac{3}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2)\\
&\quad+\frac{\bar{z}_{12}}{z_{12}} B\left(\Delta_1+\frac{1}{2},\Delta_2+\frac{3}{2}\right)\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2)\Big]
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
\mathcal{O}^{abrs}_{\Delta_1,0}(z_1,\bar{z}_1)\,\mathcal{O}^{cdtu}_{\Delta_2,0}(z_2,\bar{z}_2) &\sim \varepsilon^{abcd}\varepsilon^{rstu} B(\Delta_1+1,\Delta_2+1)\Big[\frac{z_{12}}{\bar{z}_{12}}\mathcal{O}_{\Delta_1+\Delta_2,+2}(z_2,\bar{z}_2)\\
&\quad+\frac{\bar{z}_{12}}{z_{12}}\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2)\Big]
\end{aligned} \tag{C.4}$$

C.1.2 Different spins

$$\begin{aligned}
&\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}^{A}_{\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2-\frac{1}{2}\right)\mathcal{O}^{A}_{\Delta_1+\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2)\\
&\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}_{A;\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{5}{2}\right)\mathcal{O}_{A;\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2)
\end{aligned}\tag{C.5}$$

$$\begin{aligned}
&\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}^{AB}_{\Delta_2,+1}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2)\,\mathcal{O}^{AB}_{\Delta_1+\Delta_2,+1}(z_2,\bar{z}_2)\\
&\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}_{AB;\Delta_2,-1}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+2)\,\mathcal{O}_{AB;\,\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2)
\end{aligned}\tag{C.6}$$

$$\begin{aligned}
&\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}^{abr}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{1}{2}\right)\mathcal{O}^{abr}_{\Delta_1+\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\\
&\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}^{abc}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{1}{2}\right)\mathcal{O}^{abc}_{\Delta_1+\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\\
&\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}_{abc;\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)\sim-\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-1,\Delta_2+\frac{3}{2}\right)\mathcal{O}_{abc;\,\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)
\end{aligned}\tag{C.7}$$

$$\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}^{ABCD}_{\Delta_2,0}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2+1)\,\mathcal{O}^{ABCD}_{\Delta_1+\Delta_2,0}(z_2,\bar{z}_2)\tag{C.8}$$

$$\begin{aligned}
&\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{BC}_{\Delta_2,+1}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2\right)\mathcal{O}^{ABC}_{\Delta_1+\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\\
&\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}_{BC;\,\Delta_2,-1}(z_2,\bar{z}_2)\sim 2!\,\delta^{A}_{[B}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2\right)\mathcal{O}_{C];\,\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2)
\end{aligned}\tag{C.9}$$

$$\begin{aligned}
&\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{BCD}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\sim\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+\frac{1}{2}\right)\mathcal{O}^{ABCD}_{\Delta_1+\Delta_2,0}(z_2,\bar{z}_2)\\
&\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}_{BCD\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)\sim 3\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+\frac{3}{2}\right)\delta^{A}_{[B}\mathcal{O}_{CD];\,\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2)
\end{aligned}\tag{C.10}$$

$$\begin{aligned}
&\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{BCDE}_{\Delta_2,0}(z_2,\bar{z}_2)\sim-\frac{1}{6}\varepsilon^{ABCDEFGH}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{FGH;\,\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)\\
&\mathcal{O}^{A}_{\Delta_1,+\frac{3}{2}}(z_1,\bar{z}_1)\,\mathcal{O}_{BCDE\Delta_2,0}(z_2,\bar{z}_2)\sim 3!\delta^{A}_{[B}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1-\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{CDE];\,\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)
\end{aligned}\tag{C.11}$$

$$\begin{aligned}
&\mathcal{O}^{ab}_{\Delta_1,+1}(z_1,\bar{z}_1)\,\mathcal{O}^{cdr}_{\Delta_2,+\frac{1}{2}}(z_2,\bar{z}_2)\sim\frac{1}{3!}\varepsilon^{rstu}\varepsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1,\Delta_2+\frac{1}{2}\right)\mathcal{O}_{stu;\,\Delta_1+\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)\\
&\mathcal{O}^{AB}_{\Delta_1,+1}(z_1,\bar{z}_1)\,\mathcal{O}_{CDE;\,\Delta_2,-\frac{1}{2}}(z_2,\bar{z}_2)\sim-\delta^{AB}_{CD}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1,\Delta_2+\frac{3}{2}\right)\mathcal{O}_{E;\,\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2)
\end{aligned}\tag{C.12}$$

$$\begin{aligned}
&\mathcal{O}^{ab}_{\Delta_1,+1}(z_1,\bar{z}_1)\,\mathcal{O}^{cdrs}_{\Delta_2,0}(z_2,\bar{z}_2)\sim\varepsilon^{rstu}\varepsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1,\Delta_2+1)\,\mathcal{O}_{tu;\,\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2)\\
&\mathcal{O}^{ab}_{\Delta_1,+1}(z_1,\bar{z}_1)\,\mathcal{O}^{cdef}_{\Delta_2,0}(z_2,\bar{z}_2)\sim-\varepsilon^{cdef}\varepsilon^{abgh}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1,\Delta_2+1)\,\mathcal{O}_{gh;\,\Delta_1+\Delta_2,-1}(z_2,\bar{z}_2)
\end{aligned}\tag{C.13}$$

$$\mathcal{O}^{abr}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{cdst}_{\Delta_2,0}(z_2,\bar{z}_2) \sim \varepsilon^{rstu}\varepsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{u;\,\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2)$$
$$\mathcal{O}^{abr}_{\Delta_1,+\frac{1}{2}}(z_1,\bar{z}_1)\,\mathcal{O}^{cstu}_{\Delta_2,0}(z_2,\bar{z}_2) \sim -\varepsilon^{rstu}\varepsilon^{abcd}\frac{\bar{z}_{12}}{z_{12}}B\left(\Delta_1+\frac{1}{2},\Delta_2+1\right)\mathcal{O}_{d;\,\Delta_1+\Delta_2,-\frac{3}{2}}(z_2,\bar{z}_2) \tag{C.14}$$

Similarly, all other OPEs can be extracted from the amplitudes given in [392].

C.2 Soft graviton and soft gravitino operators

We will use the soft limit of superamplitude and then perform an expansion in the Grassmann odd coordinate of the superspace to obtain the soft graviton and gravitino limits in an amplitude.

The leading and sub-leading soft factors in a superamplitude corresponding to ω_p^{-1} and ω_p^{0} were calculated in [392] using double copy relations. Here we only present relevant results and refer the readers to [392] for further details. In the celestial basis, the leading soft factor is given by

$$\mathcal{M}_N(\cdots,j-1,j,j+1,\cdots) \xrightarrow{\omega_j\to 0} \frac{1}{\omega_j}\sum_{\substack{i=1\\ i\neq j}}^{N}\omega_i\, z_{ji}\,\bar{z}_{ji}\left[\frac{z^2_{j-1,i}}{z^2_{j-1,j}\,z^2_{j,i}}+\frac{\bar{z}^2_{j-1,i}}{\bar{z}^2_{j-1}\,\bar{z}^2_{j,i}}\delta^4\left(\eta^j\right)\right]$$
$$\times\mathcal{M}_{N-1}(\cdots,j-1,j+1,\cdots) \tag{C.15}$$

One can now get the soft limit in terms of celestial superamplitude in a straightforward way. We have

$$\left\langle\prod_{n=1}^{N}\lim_{\Delta_j\to 1}(\Delta_j-1)\mathcal{O}_{\Delta_n}(z_n,\bar{z}_n,\eta^n)\right\rangle = \left(\prod_{\substack{n=1\\ n\neq j}}^{N}\int d\omega_n\,\omega_n^{\Delta_n-1}\right)\lim_{\Delta_j\to 1}\int_0^\infty d\omega_j(\Delta_j-1)\omega_j^{\Delta_j-1}$$
$$\times\,\delta^{(4)}\left(\sum_{\substack{k=1\\ k\neq j}}^{N}\omega_k q_k\right)\mathcal{M}_N(1,\ldots,n,\ldots,N)$$
$$=\left(\prod_{\substack{n=1\\ n\neq j}}^{N}\int d\omega_n\,\omega_n^{\Delta_n-1}\right)\int_0^\infty d\omega_j\frac{d}{d\omega_j}\left(\lim_{\Delta_j\to 1}\omega_j^{\Delta_j-1}\right)\delta^{(4)}\left(\sum_{\substack{k=1\\ k\neq j}}^{N}\omega_k q_k\right)\omega_j\mathcal{M}_N(1,\ldots,n,\ldots,N) \tag{C.16}$$

Using the fact that

$$\frac{d}{d\omega_j}\left(\lim_{\Delta_j\to 1}\omega_j^{\Delta_j-1}\right)=\frac{d}{d\omega_j}\theta(\omega_j)=\delta(\omega_j) \tag{C.17}$$

where $\theta(\omega)$ is the Heaviside step function, we see that the integral on ω_j on the right-hand side gives us

$$\left\langle\prod_{n=1}^{N}\lim_{\Delta_j\to 1}(\Delta_j-1)\mathcal{O}_{\Delta_n}(z_n,\bar{z}_n,\eta^n)\right\rangle = \left(\prod_{\substack{n=1\\ n\neq j}}^{N}\int d\omega_n\,\omega_n^{\Delta_n-1}\right)\delta^{(4)}\left(\sum_{\substack{k=1\\ k\neq j}}^{N}\omega_k q_k\right)$$
$$\times\lim_{\omega_j\to 0}\omega_j\mathcal{M}_N(1,\ldots,n,\ldots,N) \tag{C.18}$$

Using the soft limit (C.15) we get

$$\left\langle \prod_{n=1}^{N} \lim_{\Delta_j\to 1} (\Delta_j - 1)\mathcal{O}_{\Delta_n}(z_n,\bar{z}_n,\eta^n) \right\rangle = \sum_{\substack{i=1\\ i\neq j}}^{N} \omega_i \left\{ \frac{z_{j-1,i}^2 \bar{z}_{ji}}{z_{j-1,j}^2 z_{ji}} + \frac{\bar{z}_{j-1,i}^2 z_{ji}}{\bar{z}_{j-1,j}^2 \bar{z}_{ji}} \delta^4(\eta^j) \right\}$$
$$\times \left(\prod_{\substack{n=1\\ n\neq j}}^{N} \int_0^\infty d\omega_k \omega_k^{\Delta_k - 1} \right) \delta^{(4)}\left(\sum_{\substack{k=1\\ k\neq j}}^{N} \omega_k q_k \right) \mathcal{M}_{N-1}(1,\ldots,i,\ldots,N)$$
$$= \sum_{\substack{i=1\\ i\neq j}}^{n} \left\{ \frac{z_{j-1,i}^2 \bar{z}_{ji}}{z_{j-1,j}^2 z_{ji}} + \frac{\bar{z}_{j-1,i}^2 z_{ji}}{\bar{z}_{j-1,j}^2 \bar{z}_{ji}} \delta^8(\eta^j) \right\} \left[\prod_{\substack{n=1\\ n\neq i}}^{N} \int_0^\infty d\omega_k\, \omega_k^{\Delta_k-1} \int_0^\infty d\omega_i\, \omega_i^{\Delta_i} \right.$$
$$\left. \delta^{(4)}\left(\sum_{\substack{k=1\\ k\neq j}}^{N} \omega_k q_k \right) \mathcal{M}_{N-1}(1,\ldots,i,\ldots,N) \right]$$
$$= \sum_{\substack{i=1\\ i\neq j}}^{n} \left\{ \frac{z_{j-1,i}^2 \bar{z}_{ji}}{z_{j-1,j}^2 z_{ji}} + \frac{\bar{z}_{j-1,i}^2 z_{ji}}{\bar{z}_{j-1,j}^2 \bar{z}_{ji}} \delta^8(\eta^j) \right\} \langle \mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta_1),\cdots \mathcal{O}_{\Delta_i+1}(z_i,\bar{z}_i,\eta^i),\cdots\rangle$$

The Super-Ward identity that we get from the conformally supersoft theorem is

$$\begin{aligned} &\langle J_1(z,\bar{z},\eta)\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1)\cdots\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle \\ &= \sum_{i=1}^{N} \left\{ \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)} \frac{(z_N - z_i)^2}{(z_N - z)^2} + \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)} \frac{(\bar{z}_N - \bar{z}_i)^2}{(\bar{z}_N - \bar{z})^2} \delta^8(\eta) \right\} \\ &\qquad \times \langle \mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1),\cdots \mathcal{O}_{\Delta_i+1}(z_i,\bar{z}_i,\eta^i),\cdots,\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle \end{aligned} \tag{C.19}$$

where

$$J_1(z,\bar{z},\eta) = \lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_\Delta(z,\bar{z},\eta)$$

is the $\Delta \to 1$ soft operator. In the above soft factor, we chose the reference vector for polarization of the soft particle to be the momentum vector of nth particle. We leave this reference vector arbitrary which corresponds to a point $\xi \in \mathcal{CS}^2$. The super-Ward identity then takes the form

$$\begin{aligned} &\langle J_1(z,\bar{z},\eta)\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1)\cdots\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle \\ &= \sum_{i=1}^{N} \left\{ \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)} \frac{(\xi - z_i)^2}{(\xi - z)^2} + \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)} \frac{(\bar{\xi} - \bar{z}_i)^2}{(\bar{\xi} - \bar{z})^2} \delta^8(\eta) \right\} \\ &\qquad \times \langle \mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1),\cdots \mathcal{O}_{\Delta_i+1}(z_i,\bar{z}_i,\eta^i),\cdots,\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle \end{aligned} \tag{C.20}$$

When we expand both sides in the Grassmann variables η^i and compare coefficients, we get the Ward identity for the soft graviton operator:

$$\begin{aligned} \left\langle J_1(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)} \frac{(\xi - z_i)^2}{(\xi - z)^2} \langle \mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1), \\ \cdots \mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i),\cdots,\mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle \end{aligned} \tag{C.21}$$

and

$$\left\langle \bar{J}_1(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)} \frac{\left(\bar{\xi}-\bar{z}_i\right)^2}{\left(\bar{\xi}-\bar{z}\right)^2} \langle \mathcal{O}_{\Delta_1,\ell_1}(z_1,\bar{z}_1), \cdots \mathcal{O}_{\Delta_i+1,\ell_i}(z_i,\bar{z}_i), \cdots, \mathcal{O}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle \tag{C.22}$$

where

$$J_1(z,\bar{z}) = \lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_{\Delta,+2}(z,\bar{z}), \quad \bar{J}_1(z,\bar{z}) = \lim_{\Delta\to 1}(\Delta-1)\mathcal{O}_{\Delta,-2}(z,\bar{z}) \tag{C.23}$$

are the $\Delta = 1$ soft graviton operators. The subleading soft factor was also calculated in [392]. It turns out that it is the same as the subleading soft factor for positive and negative helicity graviton in pure gravity [113, Eq. (2.9)]. We then write the super Ward identity following the calculations in [354]:

$$\begin{aligned}
&\langle J_0(z,\bar{z},\eta)\mathcal{O}_{\Delta_1}(z,\bar{z},\eta^1)\cdots\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle \\
&= \sum_{i=1}^{N}\left\{\frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)}{(\xi-z)}((\bar{z}-\bar{z}_i)\partial_{\bar{z}_i}-2\bar{h}_i)+\frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)}{(\bar{\xi}-\bar{z})}\delta^8(\eta)((z-z_i)\partial_{z_i}-2h_i)\right\} \\
&\qquad \times\langle\mathcal{O}_{\Delta_1}(z_1,\bar{z}_1,\eta^1),\cdots\mathcal{O}_{\Delta_i}(z_i,\bar{z}_i,\eta^i)\cdots,\mathcal{O}_{\Delta_N}(z_N,\bar{z}_N,\eta^N)\rangle
\end{aligned} \tag{C.24}$$

where

$$J_0(z,\bar{z},\eta) = \lim_{\Delta\to 0}\Delta[\mathcal{O}_{\Delta,+2}(z,\bar{z})+\delta^8(\eta)\mathcal{O}_{\Delta,-2}(z,\bar{z})].$$

only contains the $\Delta = 0$ soft graviton operators. This immediately gives us the subleading soft graviton limit:

$$\left\langle J_0(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)}\frac{(\xi-z_i)}{(\xi-z)}((\bar{z}-\bar{z}_i)\partial_{\bar{z}_i}-2\bar{h}_i) \times\langle\cdots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\rangle \tag{C.25}$$

and

$$\left\langle \bar{J}_0(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)}\frac{(\bar{\xi}-\bar{z}_i)}{(\bar{\xi}-\bar{z})}((z-z_i)\partial_{z_i}-2h_i) \times\langle\cdots\mathcal{O}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\rangle \tag{C.26}$$

where

$$J_0(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}_{\Delta,+2}(z,\bar{z}), \quad \bar{J}_0(z,\bar{z}) = \lim_{\Delta\to 0}\Delta\mathcal{O}_{\Delta,-2}(z,\bar{z}) \tag{C.27}$$

are the $\Delta = 0$ soft graviton operators and $h_i = \frac{\Delta_i+\ell_i}{2}, \bar{h}_i = \frac{\Delta_i-\ell_i}{2}$ are the conformal weights of the operator $\mathcal{O}_{\Delta_i,\ell_i}(z,\bar{z})$.

Next, we move on to the soft gravitino operator. The leading soft gravitino limit for super-amplitudes is given by

$$\mathcal{M}_{N+1}(\psi^A_{s+},\{p_1,\eta^1\},\ldots\{p_N,\eta^N\}) = \sum_{i=1}^{N}\frac{[si]\langle ri\rangle}{\langle si\rangle\langle rs\rangle}\frac{\partial}{\partial\eta_{iA}}\mathcal{M}_N(\{p_1,\eta^1\},\ldots\{p_N,\eta^N\}), \tag{C.28}$$

where r is the reference vector corresponding to point $\xi \in \mathcal{CS}^2$. The negative helicity soft gravitino limit can be obtained by conjugating the soft factor. We can expand both sides in η^i and get the soft theorem in terms of component fields. Note that because of $\partial/\partial\eta_{iA}$, the soft gravitino operator changes the spin of the particle $\ell_i \to \ell_i^c \equiv \ell_i - \frac{1}{2}$. Thus we can only have

$$\ell_i \in \{-3/2, -1, -1/2, 0, +1/2, +1, +3/2, +2\}. \tag{C.29}$$

For negative helicity gravitino $\ell_i^c \to \ell_i$ and clearly

$$\ell_i^c \in \{-2, -3/2, -1, -1/2, 0, +1/2, +1, +3/2\}. \tag{C.30}$$

The explicit soft theorem in terms of celestial amplitudes is given by

$$\left\langle J^A_{1/2}(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} f(A,\ell_i,*_i,*_i')(-1)^{\sigma_i} \frac{(\bar{z}-\bar{z}_i)}{(z-z_i)} \frac{(\xi - z_i)}{(\xi - z)} \left\langle \cdots \mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell_i^c}(z_i,\bar{z}_i), \cdots \right\rangle \tag{C.31}$$

and

$$\left\langle \bar{J}_{1/2\,A}(z,\bar{z}) \prod_{n=1}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n^c}(z_n,\bar{z}_n) \right\rangle = \sum_{i=1}^{N} \bar{f}(A,\ell_i^c,*_i,*_i')(-1)^{\sigma_i} \frac{(z-z_i)}{(\bar{z}-\bar{z}_i)} \frac{(\bar{\xi} - \bar{z}_i)}{(\bar{\xi} - \bar{z})} \left\langle \cdots \mathcal{O}^{*_i'}_{\Delta_i+\frac{1}{2},\ell_i}(z_i,\bar{z}_i), \cdots \right\rangle \tag{C.32}$$

where

$$J^A_{1/2}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}} \left(\Delta - \frac{1}{2}\right) \mathcal{O}^A_{\Delta,+\frac{3}{2}}(z,\bar{z}), \quad \bar{J}_{1/2\,A}(z,\bar{z}) = \lim_{\Delta\to\frac{1}{2}} \left(\Delta - \frac{1}{2}\right) \mathcal{O}_{\Delta,-\frac{3}{2},A}(z,\bar{z}) \tag{C.33}$$

are the soft gravitino operators.

C.3 OPE of the composite current $\mathcal{G}^{CD}_{AB}(z,\bar{z})$

We begin by calculating the OPEs $G\overline{G}$. We have[1]

$$\begin{aligned} &\left\langle G_{AB}(z,\bar{z}) \overline{G}^{CD}(w,\bar{w}) \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \\ &= \lim_{\substack{\Delta_1\to 0\\ \Delta_2\to 0}} \frac{\Delta_1\Delta_2}{\pi^2} \int d^2z_1 \frac{1}{(z-z_1)^a} \frac{1}{(\bar{z}-\bar{z}_1)^b} \frac{1}{(\bar{w}-\bar{z}_2)^{a'}} \frac{1}{(w-z_2)^{b'}} \\ &\quad \times \left\langle \mathcal{O}_{AB\;\Delta_1,-1}(z_1,\bar{z}_1) \mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2) \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n) \right\rangle \end{aligned} \tag{C.34}$$

[1]Since there is also overlap as there in case of soft gravitino currents in section 5.3.1 we do not separate the operators in the correlator according to their spins ℓ, ℓ' and keep the spins to be arbitrary here as well.

By taking the soft limit of the first operator $\Delta_1 \to 0$,

$$\begin{aligned}
&\left\langle G_{AB}(z)\overline{G}^{CD}(\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&= \lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi^2}\int d^2z_1\int d^2z_2\frac{1}{(z-z_1)^a}\frac{1}{(\bar{z}-\bar{z}_1)^b}\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}} \\
&\times\Bigg[-\delta_{AB}^{CD}\frac{z_1-z_2}{\bar{z}_1-\bar{z}_2}\left\langle\mathcal{O}_{\Delta_2,+2}(z_2,\bar{z}_2)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&+\sum_{j=3}^{N}f(A,B,\ell_j,*_j,*_j')\frac{z_1-z_j}{\bar{z}_1-\bar{z}_j}\left\langle\mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2)\cdots\mathcal{O}^{*_j'}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\Bigg]
\end{aligned} \tag{C.35}$$

Now doing the first integral using (5.61), we get

$$\begin{aligned}
&\left\langle G_{AB}(z)\overline{G}^{CD}(\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&= \lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}} \\
&\times\Bigg[-\delta_{AB}^{CD}C_1(b,a)\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b}\left\langle\mathcal{O}_{\Delta_2,+2}(z_2,\bar{z}_2)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&+\sum_{j=3}^{N}f(A,B,\ell_j,*_j,*_j')C_1(b,a)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar{z}_j-\bar{z})^b} \\
&\qquad\times\left\langle\mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2)\cdots\mathcal{O}^{*_j'}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\Bigg]
\end{aligned} \tag{C.36}$$

We now use the collinear limits of the graviton operator with other fields in the first term and take the conformally soft limit $\Delta_2 \to 0$ in the second term.

The first term becomes

$$\begin{aligned}
&\lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\int d^2z_2 \frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\Bigg[-\delta_{AB}^{CD}C_1(b,a)\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b}\\
&\qquad\qquad \times\left\langle \mathcal{O}_{\Delta_2,+2}(z_2,\bar{z}_2)\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle\Bigg]\\
&=-\delta_{AB}^{CD}C_1(b,a)\lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b}\\
&\times\sum_{i=3}^{N}B(\Delta_2-1,f(\Delta_i))\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\\
&=-\delta_{AB}^{CD}C_1(b,a)\frac{1}{\pi}\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^b}\\
&\times\sum_{i=3}^{N}(1-f(\Delta_i))\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}\times\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle
\end{aligned} \tag{C.37}$$

where we used

$$\lim_{\Delta_2\to 0}\Delta_2\, B(\Delta_2-1,f(\Delta_i)) = 1-f(\Delta_i)$$

In the second term in (C.36) now we can take $\Delta_2 \to 0$ limit,

$$
\begin{aligned}
&\lim_{\Delta_2\to 0}\frac{\Delta_2}{\pi}\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar{z}_j-\bar{z})^{b}}\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}} \\
&\qquad\qquad\times\left\langle \mathcal{O}^{CD}_{\Delta_2,+1}(z_2,\bar{z}_2)\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&=\frac{1}{\pi}\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j)C_1(b,a)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar{z}_j-\bar{z})^{b}} \\
&\times\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{\bar{z}_2-\bar{z}_j}{z_2-z_j}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&+\frac{1}{\pi}\sum_{\substack{i,j=3\\ i\neq j}}^{N} f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_i,*_i,*'_i)C_1(b,a)\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar{z}_j-\bar{z})^{b}} \\
&\times\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}\left\langle \cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&=\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j)C_1(b',a')\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar{z}_j-\bar{z})^{b}} \\
&\times\frac{1}{(z_j-w)^{b'}}\frac{1}{(\bar{z}_j-\bar{w})^{a'-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&+\sum_{\substack{i,j=3\\ i\neq j}}^{N} f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_i,*_i,*'_i)C_1(b,a)C_1(b',a')\frac{1}{(z_j-z)^{a-2}}\frac{1}{(\bar{z}_j-\bar{z})^{b}} \\
&\times\frac{1}{(z_i-w)^{b'}}\frac{1}{(\bar{z}_i-\bar{w})^{a'-2}}\left\langle \cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle
\end{aligned}
\tag{C.38}
$$

Combining the two integrals we get

$$\begin{aligned}
&\left\langle G_{AB}(z,\bar{z})\overline{G}^{CD}(w,\bar{w})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&= -\delta^{CD}_{AB}C_1(b,a)\frac{1}{\pi}\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{1}{(z_2-z)^{a-2}(\bar{z}_2-\bar{z})^{b}} \\
&\times\sum_{i=3}^{N}(1-f(\Delta_i))\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}\times\left\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&+\sum_{j=3}^{N}f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j)C_1(b',a')\frac{1}{\left(z_j-z\right)^{a-2}}\frac{1}{\left(\bar{z}_j-\bar{z}\right)^{b}} \\
&\times\frac{1}{\left(z_j-w\right)^{b'}}\frac{1}{\left(\bar{z}_j-\bar{w}\right)^{a'-2}}\left\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&+\sum_{\substack{i,j=3\\ i\neq j}}^{N}f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_i,*_i,*'_i)C_1(b,a)C_1(b',a')\frac{1}{\left(z_j-z\right)^{a-2}}\frac{1}{\left(\bar{z}_j-\bar{z}\right)^{b}} \\
&\times\frac{1}{(z_i-w)^{b'}}\frac{1}{(\bar{z}_i-\bar{w})^{a'-2}}\left\langle\cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle
\end{aligned} \tag{C.39}$$

Here when we take the normal order of this composite current we only need to care about the non-singular terms in the above OPE. The non-singular term in the integral above can be obtained by taking $z\to w$ limit in the integral. The integral can then be evaluated as

$$\begin{aligned}
&\int d^2z_2\frac{1}{(\bar{w}-\bar{z}_2)^{a'}}\frac{1}{(w-z_2)^{b'}}\frac{1}{(z_2-z)^{a-2}}\frac{1}{(\bar{z}_2-\bar{z})^{b}}\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i}\quad\xrightarrow{z=w} \\
&\qquad=(-1)^{a+b-2}\int d^2z_2\frac{1}{(\bar{z}-\bar{z}_2)^{a'+b}}\frac{1}{(z-z_2)^{b'+a-2}}\frac{\bar{z}_2-\bar{z}_i}{z_2-z_i} \\
&\qquad=(-1)^{a+b}C_1\left(a+b'-2,a'+b\right)\frac{1}{(z_i-z)^{a'+b-2}}\frac{1}{(\bar{z}_i-\bar{z})^{a+b'-2}}
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\langle : G_{AB}(z,\bar{z})\overline{G}^{CD}(z,\bar{z}) : \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&= \frac{(-1)^{a+b+1}}{\pi}\delta_{AB}^{CD} C_1(b,a) C\left(a+b'-2,a'+b\right)\sum_{i=3}^{N}(1-f(\Delta_i)) \\
&\qquad\times \frac{1}{(z_i-z)^{a'+b-2}}\frac{1}{(\bar{z}_i-\bar{z})^{a+b'-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell'_N}(z_N,\bar{z}_N)\right\rangle \\
&+\sum_{j=3}^{N} f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j)C_1(b,a)C_1(b',a')(-1)^{a+b+a'+b'} \\
&\qquad\times \frac{1}{\left(z-z_j\right)^{a+b'-2}}\frac{1}{\left(\bar{z}-\bar{z}_j\right)^{a'+b-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&+\sum_{\substack{i,j=3\\ i\neq j}}^{N} f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_i,*_i,*'_i)C_1(b,a)C_1(b',a')(-1)^{a+b+a'+b'}\frac{1}{\left(z-z_j\right)^{a-2}}\frac{1}{\left(\bar{z}-\bar{z}_j\right)^{b}} \\
&\qquad\times \frac{1}{\left(z-z_i\right)^{b'}}\frac{1}{\left(\bar{z}-\bar{z}_i\right)^{a'-2}}\left\langle \cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j+1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i-1}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle
\end{aligned} \tag{C.40}$$

Similarly, we have,

$$\begin{aligned}
&\left\langle : \overline{G}^{CD}(z,\bar{z})G_{AB}(z,\bar{z}) : \prod_{n=3}^{N} \mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&= \frac{(-1)^{a'+b'+1}}{\pi}\delta_{AB}^{CD} C_1(b',a') C\left(a'+b-2,a+b'\right)\sum_{i=3}^{N}(1-f(\Delta_i)) \\
&\qquad\times \frac{1}{(z_i-z)^{a+b'-2}}\frac{1}{(\bar{z}_i-\bar{z})^{a'+b-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&+\sum_{j=3}^{N} \bar{f}(C,D,\ell_j,*_j,*'_j)f(A,B,\ell_j-1,*'_j,*''_j)C_1(b',a')C_1(b,a)(-1)^{a+b+a'+b'} \\
&\qquad\times \frac{1}{\left(z-z_j\right)^{a'+b-2}}\frac{1}{\left(\bar{z}-\bar{z}_j\right)^{a+b'-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle \\
&+\sum_{\substack{i,j=3\\ i\neq j}}^{N} \bar{f}(C,D,\ell_j,*_j,*'_j)f(A,B,\ell_i,*_i,*'_i)C_1(b,a)C_1(b',a')(-1)^{a+b+a'+b'}\frac{1}{\left(\bar{z}-\bar{z}_j\right)^{a'-2}}\frac{1}{\left(z-z_j\right)^{b'}} \\
&\qquad\times \frac{1}{\left(\bar{z}-\bar{z}_i\right)^{b}}\frac{1}{\left(z-z_i\right)^{a-2}}\left\langle \cdots\mathcal{O}^{*'_j}_{\Delta_j,\ell_j-1}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*'_i}_{\Delta_i,\ell_i+1}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle
\end{aligned} \tag{C.41}$$

We have the correlator of the normalized current with any conformal primary as,

$$\begin{aligned}
&\left\langle \mathcal{G}_{AB}^{CD}(z,\bar{z})\prod_{n=3}^{N}\mathcal{O}^{*_n}_{\Delta_n,\ell_n}(z_n,\bar{z}_n)\right\rangle \\
&= -\delta_{AB}^{CD}\sum_{i=3}^{N}(1-f(\Delta_i))\left[\frac{(-1)^{a+b}}{\pi}C_1(b,a)C\left(a+b'-2,a'+b\right)\frac{1}{(z_i-z)^{a'+b-2}}\frac{1}{(\bar{z}_i-\bar{z})^{a+b'-2}}\right.\\
&\qquad\times\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\\
&-\frac{(-1)^{a'+b'}}{\pi}C_1(b',a')C_1\left(a'+b-2,a+b'\right)\frac{1}{(z_i-z)^{a+b'-2}}\frac{1}{(\bar{z}_i-\bar{z})^{a'+b-2}}\\
&\qquad\left.\times\langle\mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*_i}_{\Delta_i,\ell_i}(z_i,\bar{z}_i)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\rangle\right]\\
&+(-1)^{a+b+a'+b'}C_1(b,a)C_1(b',a')\left[f(A,B,\ell_j,*_j,*'_j)\bar{f}(C,D,\ell_j+1,*'_j,*''_j)\right.\\
&\quad\times\frac{1}{\left(z-z_j\right)^{a+b'-2}}\frac{1}{\left(\bar{z}-\bar{z}_j\right)^{a'+b-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\\
&\qquad-\bar{f}(C,D,\ell_j,*_j,*'_j)f(A,B,\ell_j-1,*'_j,*''_j)\\
&\quad\left.\times\frac{1}{\left(z-z_j\right)^{a'+b-2}}\frac{1}{\left(\bar{z}-\bar{z}_j\right)^{a+b'-2}}\left\langle \mathcal{O}^{*_3}_{\Delta_3,\ell_3}(z_3,\bar{z}_3)\cdots\mathcal{O}^{*''_j}_{\Delta_j,\ell_j}(z_j,\bar{z}_j)\cdots\mathcal{O}^{*_N}_{\Delta_N,\ell_N}(z_N,\bar{z}_N)\right\rangle\right]
\end{aligned} \tag{C.42}$$

The last term in both (C.40) and (C.41) cancels when we take the commutator. Now in the above OPE, we can see that the first term which has the graviton soft limits, does not satisfy our requirement explained in (5.58). Hence we require that the two terms in the first expression be the same so that they cancel once we take the commutator. This is equivalent to the requirement

$$C_1\left(b',a'\right)C_1\left(a'+b-2,a+b'\right)(-1)^{a+b}=C_1(b,a)C_1\left(a+b'-2,a'+b\right)(-1)^{a'+b'} \tag{C.43}$$

and

$$a'+b-2=a+b'-2 \tag{C.44}$$

Now in (C.43) by substituting the explicit expression from (5.62) we have

$$\left(-a'-b+1\right)\left(-a'-b+2\right)(-a+1)(-a+2)=\left(-a'+1\right)\left(-a'+2\right)\left(-a-b'+1\right)\left(-a-b'+2\right).$$

which after using (C.44) gives

$$(-a+1)(-a+2)=\left(-a'+1\right)\left(-a'+2\right) \tag{C.45}$$

which clearly has solutions. Hence correlator corresponding to this normal order current is,

$$
\begin{aligned}
&\left\langle \mathcal{G}_{AB}^{CD}(z,\bar{z}) \prod_{n=3}^{N} \mathcal{O}_{\Delta_n,\ell_n}^{*_n}(z_n,\bar{z}_n) \right\rangle \\
&= (-1)^{a+b+a'+b'} C_1(b,a) C_1(b',a') \Bigg[f(A,B,\ell_j,*_j,*_j') \bar{f}(C,D,\ell_j+1,*_j',*_j'') \\
&\qquad \times \frac{1}{\left(z-z_j\right)^{a+b'-2}} \frac{1}{\left(\bar{z}-\bar{z}_j\right)^{a'+b-2}} \left\langle \mathcal{O}_{\Delta_3,\ell_3}^{*_3}(z_3,\bar{z}_3) \cdots \mathcal{O}_{\Delta_j,\ell_j}^{*_j''}(z_j,\bar{z}_j) \cdots \mathcal{O}_{\Delta_N,\ell_N}^{*_N}(z_N,\bar{z}_N) \right\rangle \\
&\qquad\quad - \bar{f}(C,D,\ell_j,*_j,*_j') f(A,B,\ell_j-1,*_j',*_j'') \\
&\qquad \times \frac{1}{\left(z-z_j\right)^{a'+b-2}} \frac{1}{\left(\bar{z}-\bar{z}_j\right)^{a+b'-2}} \left\langle \mathcal{O}_{\Delta_3,\ell_3}^{*_3}(z_3,\bar{z}_3) \cdots \mathcal{O}_{\Delta_j,\ell_j}^{*_j''}(z_j,\bar{z}_j) \cdots \mathcal{O}_{\Delta_N,\ell_N}^{*_N}(z_N,\bar{z}_N) \right\rangle \Bigg]
\end{aligned}
\tag{C.46}
$$

APPENDIX D
SOME REQUIREMENTS FOR ADS SPACETIMES

D.1 Some requirements to work in AdS

The AdS_{d+1} space is described by embedding it in $d+2$ dimensional Minkowski space described by $(\mathbb{R}^{d,2}, g_{ab})$. The coordinates of AdS_{d+1} of length R is a set of points $X \equiv (X^0, X^1, \cdots, X^d)$. The spacetime interval is given by,

$$ds^2 = -\left(dX^0\right)^2 + \left(dX^1\right)^2 + \cdots + \left(dX^{d-1}\right)^2 - \left(dX^d\right)^2$$

We eliminate X^d using the equation of the space to arrive at the metric. Let $W = X^d$ and $X^2 = X.X = \eta_{ab}X^aX^b; a = 0,1,\ldots,d-2,d-1, \eta_{ab}$ follows the mostly positive signature [421]. Hence, $W^2 = R^2 + X^2$, which implies,

$$\begin{aligned} ds^2 &= \eta_{ab}dX^a dX^b - dW^2 \\ \therefore ds^2 &= \left(\eta_{ab} - \frac{\eta_{a\lambda}\eta_{b\rho}X^\lambda X^\rho}{R^2+X^2}\right)dX^a dX^b \end{aligned} \tag{D.1}$$

So, the metric and inverse metric for AdS space are,

$$g_{ab}(X) = \eta_{ab} - \frac{\eta_{a\lambda}\eta_{b\rho}X^\lambda X^\rho}{R^2+X^2}, \quad g^{ab}(X) = \eta^{ab} + \frac{\eta_{a\lambda}\eta_{b\rho}X^\lambda X^\rho}{R^2} \tag{D.2}$$

We have the local coordinates X defined on the bulk of AdS. The covariant derivatives and the metric in this space are given by

$$\nabla_a = \partial_a - \Gamma_a, \quad \Gamma^c_{ab} = -\frac{1}{\mathrm{R}^2}X^c g_{ab}(X). \tag{D.3}$$

All of the above equations are exact in R.

D.2 AdS Casimir in the flat space limit

In this appendix, we will explicitly see that the AdS Symmetry Group (SO$(d,2)$) reduces to the Poincaré group in the flat space limit of AdS ($R \to \infty$). As a consequence of this, one can also see that under this limit, the eigenvalue of the AdS Casimir can be written as the sum of the eigenvalues of the two Casimirs of the Poincaré group, namely, spin and momentum.

To prove that the AdS symmetry group reduces to the Poincaré group, we will look at the algebra of the generators of a Special Orthogonal group and then use the symmetry generators we have defined in our formalism to show that indeed the SO$(d,2)$ group reduces to Poincaré group in $d+1$ dimensions. We can start with identifying the SO$(d+1)$ part of the Poincaré group. Consider the symmetry generators we defined earlier in Eq.(6.1),

$$M_{AB} = \mathcal{X}_A\partial_B - \mathcal{X}_B\partial_A$$

Using the definition $\mathcal{X} = (X_0,\ldots,X_d,R)$ and $\mathcal{P} = (P_0,\ldots,P_d,0)$ for $a,b = 0,1,\ldots,d$, we have,

$$\mathrm{M}_{ab} = X_a\partial_b - X_b\partial_a.$$

Clearly, for the $d+1$ dimensional vector X_a, M_{ab} are the position space generators of the group $\mathrm{SO}(d+1)$.

We can further investigate the momentum space generators, which can be written as

$$M_{ab} = P_a \frac{\partial}{\partial P^b} - P_b \frac{\partial}{\partial P^a}, \quad M_{a,d+1} = P_a \frac{\partial}{\partial \mathcal{P}^{d+1}} - \mathcal{P}_{d+1} \frac{\partial}{\partial P^a} = iP_a \sqrt{R^2 - \frac{\partial}{\partial P} \cdot \frac{\partial}{\partial P}} \tag{D.4}$$

For large R, Taylor expanding the above expression in orders of $1/R^2$ gives,

$$M_{a,d+1} = iRP_a + \mathcal{O}(1/R^2) \tag{D.5}$$

This expression tells us that at leading order, $M_{a,d+1}$ resembles the momentum of the particle (generator of translation) up to a constant. This gets a correction at $\mathcal{O}(1/R^2)$ due to the AdS potential which changes the form of these generators such that they are no longer generators of translation. This is expected as the AdS manifold is not invariant under translation.

Let us find the algebra of the generators corresponding to $SO(d+1)$ and translations in our notation. We can start with computing $[M_{ab}, M_{c,d+1}]$, $a,b,c = 0,1,\ldots,d$.

$$[M_{ab}, M_{c,d+1}] = \left(P_a \frac{\partial}{\partial P^b} - P_b \frac{\partial}{\partial P^a}\right)\left(iP_c R \sqrt{1 - \frac{\partial_P^2}{R^2}}\right) - \left(iP_c R \sqrt{1 - \frac{\partial_P^2}{R^2}}\right)\left(P_a \frac{\partial}{\partial P^b} - P_b \frac{\partial}{\partial P^a}\right) \tag{D.6}$$

As before, we can Taylor expand the terms for large R. At leading order, we have

$$[M_{ab}, M_{c,d+1}] = i(P_a\, \eta_{bc} - iP_b\, \eta_{ac})R + \mathcal{O}(1/R^2) \tag{D.7}$$

Here, η_{ab} is the flat space metric in $d+1$ dimensions. This expression at leading order matches the commutation relations of a rotation generator J_{ab} and the translation generator K_c in flat space up to an overall negative sign.

$$[J_{ab}, K_c] = i(K_b \eta_{ac} - K_a \eta_{bc})$$

This semi-direct product structure will break down at higher orders of $1/R^2$. The consequence of this is seen in terms of the eigenvalues of the AdS and Poincaré Casimirs.

The Poincaré group has two Casimirs as mass and spin, which is a result of the structure of the group. However, the AdS symmetry group has only one physically relevant Casimir whose eigenvalue is a sum of two quantities, i.e., $M^2R^2 = \Delta(\Delta - d) + l(l+d-2)$. Here, $l(l+d-2)$ corresponds to the spin part of the Casimir in Poincaré group (this can be easily seen for $d=3$) while the other part $\Delta(\Delta - d)$ corresponds to the mass Casimir as this is the only term that survives when spin is zero.

The key observation of this discussion is that the structure of the Poincaré group allows for two separate Casimirs for mass and spin; however, if we perturb the flat space with an AdS potential, this structure breaks down, and the AdS Casimir has information about both the mass and the spin, which was expected as translation symmetry breaks down due to the AdS potential. This understanding serves as a small step in understanding the bigger question of how AdS behaves in the large R or the flat space limit and how the mathematical underpinnings can verify our approach, formalism, and understanding of the problem.

D.3 Bulk to boundary

The result presented in the main text can also be derived from a more tedious and traditional method using the "complete" equation of motion for the massive vector field as derived from the lagrangian, which is really a combined statement about both transversality and the on-shell condition.

In the first step, in terms of covariant derivatives, the EOM for massive vector field $A^{\nu}(\nu = 0,1,..,d)$ is given by,

$$\left[g_{\mu\nu}(\nabla_X^2 - M^2) - \nabla_\nu \nabla_\mu\right] A^{\nu} = 0 \tag{D.8}$$

Unlike flat space($[\partial_\mu, \partial_\nu] = 0$), $[\nabla_\mu, \nabla_\nu] \neq 0$, this commutator is precisely the Riemann curvature tensor. Here, $M^2R^2 = \Delta(\Delta - d) + (d-1)$, where $l = 1$ for the spinning vector field.

As we have seen, the AdS Casimir splits into mass and spin term in the flat space limit. Hence we can make the gauge choice, $P^2 = -m^2 = -\frac{\Delta(\Delta-d)}{R^2}$. As a result of which, we have $M^2 = m^2 + \frac{(d-1)}{R^2}$ and the propagator $\hat{\Pi}^{\mu\nu}(P;X)$ must satisfy this equation of motion for the vector field.

Note that, here, we have used the homogenous function that is dependent on $P^\mu/|P|$. Using the EOM in terms of covariant derivatives, one would compute its action on the propagator. The propagator must satisfy,

$$\left[g_{\mu\nu}\left(\nabla_X^2 - M^2\right) - \nabla_\nu \nabla_\mu\right] \hat{\Pi}^{\nu\beta} = 0 \tag{D.9}$$

The correct equation can be directly derived from first principles using the free Lagrangian for a massive vector field. The ambiguity in the order of covariant derivatives is subtle. If one had to naively promote the partial derivatives to covariant derivatives in the equation of motion for a massive vector field, they would end up with Eq.(D.9). However, in deriving the equation of motion for a massive vector field in flat space, one uses the fact that partial derivatives commute and that's how we end up with the equation of motion as we know it to be. However, covariant derivatives do not commute; hence, when deriving the equation of motion for a massive vector field in AdS, we must take care of this fact, and so, we end up with Eq.(D.9). We have

$$\begin{aligned}\nabla_\nu(\nabla_\mu \Pi^{\nu\beta}) = {} & \partial_\nu\partial_\mu \Pi^{\nu\beta} + (\partial_\nu \Pi^{\alpha\beta})\Gamma^{\nu}_{\alpha\mu} + (\partial_\nu \Gamma^{\nu}_{\alpha\mu})\Pi^{\alpha\beta} + (\partial_\nu \Pi^{\nu\alpha})\Gamma^{\beta}_{\alpha\mu} + (\partial_\nu \Gamma^{\beta})_{\alpha\mu})\Pi^{\nu\alpha} \\ & + (\partial_\mu \Pi^{\beta\sigma})\Gamma^{\nu}_{\nu\sigma} + (\partial_\mu \Pi^{\nu\sigma})\Gamma^{\beta}_{\nu\sigma} - (\partial_\rho \Pi^{\nu\beta})\Gamma^{\rho}_{\nu\mu}\end{aligned} \tag{D.10}$$

After some simplifications, we can write

$$\nabla_\nu(\nabla_\mu \Pi^{\nu\beta}) = \partial_\mu\partial_\nu \Pi^{\nu\beta} - \frac{(d+2)\eta_{\mu\rho}\Pi^{\rho\beta}}{R^2} - \frac{X_\rho \partial_\mu \Pi^{\rho\beta}}{R^2} - \frac{X^\beta \eta_{\rho\sigma}\partial_\mu \Pi^{\rho\sigma}}{R^2} - \frac{X^\beta \eta_{\rho\mu}\partial_\nu \Pi^{\nu\rho}}{R^2} \tag{D.11}$$

In the above expression, one must be careful about the last term. One can explicitly check that for the Bulk to Boundary propagator, the last terms will vanish explicitly. However, this term will be important when considering the Bulk-to-Bulk propagator equation of motion.

Substituting the expression of Eq.(D.11) in Eq.(D.9), we get

$$\begin{aligned}\left[\left(\eta_{\mu\nu}-\frac{X_\mu X_\nu}{X^2+R^2}\right)(\nabla_X^2-M^2)-\partial_\mu\partial_\nu+\frac{X_\nu\partial_\mu}{R^2}+\frac{(d+2)\eta_{\mu\nu}}{R^2}\right]\Pi^{\nu\beta}\\+\frac{X^\beta\eta_{\rho\mu}\partial_\nu\Pi^{\nu\rho}}{R^2}+\frac{X^\beta\eta_{\rho\sigma}\partial_\mu\Pi^{\rho\sigma}}{R^2}=0\end{aligned}\tag{D.12}$$

Here,

$$\begin{aligned}\nabla_X^2\Pi^{\nu\beta}=g^{\rho\sigma}\nabla_\rho\nabla_\sigma\Pi^{\nu\beta}=\partial^2\Pi^{\nu\beta}-\frac{1}{R^2}\Big[-X^\rho X^\sigma\partial_\rho\partial_\sigma\Pi^{\nu\beta}+2\Pi^{\nu\beta}+2X^\nu\partial_\alpha\Pi^{\alpha\beta}\\+2X^\beta\partial_\alpha\Pi^{\nu\alpha}-(d+1)X^\alpha\partial_\alpha\Pi^{\nu\beta}\Big]\end{aligned}$$

We will use the definition used in section 6.2 and write the scale covariant function in momentum space as a function of unit momentum vector $\hat{P}_i=\frac{P_i}{|P_i|}$, and we define the scalar $\eta=i\hat{P}.X$, and some useful relations in terms of η are,

$$\partial_\mu=i\hat{P}_\mu\partial_\eta,\quad \partial_\mu\eta=i\hat{P}_\mu,\qquad X^\mu\partial_\mu\eta=\eta,\quad P^2=-M^2,\quad |P|^2=P_aP_b\eta^{ab}.\tag{D.13}$$

Now, one can consider an ansatz order-by-order to solve the equations for the propagator as a function of unit vector $\hat{P}$,

$$\hat{\Pi}_1^{\nu\beta}=e^{iM\eta}f_1^{\nu\beta}\tag{D.14}$$

$$\hat{\Pi}_2^{\nu\beta}=\frac{1}{R^2}e^{iM\eta}f_2^{\nu\beta}\tag{D.15}$$

$$\vdots$$

$$\hat{\Pi}_{n+1}^{\nu\beta}=\frac{1}{R^{2n}}e^{iM\eta}f_{n+1}^{\nu\beta}\tag{D.16}$$

Up to subleading order, we can write,

$$\widetilde{\Pi}^{\nu\beta}(\eta)=e^{iM\eta}\left[f_1^{\nu\beta}+\frac{1}{R^2}f_2^{\nu\beta}\right]\tag{D.17}$$

The EOM from Eq.(D.12) at leading and subleading order takes the form as follows:

At leading Order:

$$\left[\eta_{\mu\nu}(\partial_X^2-M^2)-\partial_\mu\partial_\nu\right]e^{iM\eta}f_1^{\nu\beta}=0.\tag{D.18}$$

This equation matches the EOM for a massive vector field in flat space, and the solution matches the well-known result for the flat space. The solution of the leading order equation in (D.18) leads to

$$f_1^{\nu\beta}=\eta^{\nu\beta}-\hat{P}^\nu\hat{P}^\beta,\qquad \mathrm{Tr}\,f_1=d,\quad P_\nu f_1^{\nu\beta}=0\tag{D.19}$$

At subleading Order:

$$\begin{aligned}&\Big[\eta_{\mu\nu}(\partial^2-M^2)-\partial_\mu\partial_\nu\Big]e^{iM\eta}f_2^{\nu\beta}+\Big[\eta_{\mu\nu}\Big(d+(X\cdot\partial_X)^2+(d+1)X\cdot\partial_X\Big)+X_\nu\partial_\mu\Big]e^{iM\eta}f_1^{\nu\beta}\\&-2\eta_{\mu\nu}\Big(X^\nu(\partial_\alpha e^{iM\cdot\eta})f_1^{\alpha\beta}+X^\beta(\partial_\alpha e^{iM\cdot\eta})f_1^{\alpha\nu}\Big)+d\,X^\beta\partial_\mu e^{iM\cdot\eta}=0\end{aligned}\tag{D.20}$$

Using the parametrization in Eq.(D.13) we can write,

$$\begin{aligned}&\eta_{\mu\nu}\left(d-(d+1)M(X.\hat{P})+M^2(X.\hat{P})^2\right)f_1^{\nu\beta}-MX^\beta\hat{P}_\mu(d+1)+M(X.\hat{P})\hat{P}^\beta\hat{P}_\mu\\&+\eta_{\mu\nu}\Big(-2M(\hat{P}.\partial)+\partial^2\Big)f_2^{\nu\beta}-\left(\partial_\nu\partial_\mu+M^2\hat{P}_\mu\hat{P}_\nu-M\left(\hat{P}_\mu\partial_\nu+\hat{P}_\nu\partial_\mu\right)\right)f_2^{\nu\beta}=0\end{aligned}\tag{D.21}$$

Of course, this is the positive energy solution, and on physical grounds, we discard the negative energy solution.

The solution of the subleading order equation in (D.21) is given by,

$$\begin{aligned}f_2^{\nu\beta}=&\left[-\frac{(d+2)(\hat{P}\cdot X)}{4M}-\frac{d(\hat{P}\cdot X)^2}{4}+\frac{M(\hat{P}\cdot X)^3}{6}\right]f_1^{\nu\beta}\\&-\frac{(d+1)}{M}X^\beta\hat{P}^\nu+\frac{1}{M}X^\nu\hat{P}^\beta+\frac{d}{M^2}\hat{P}^\beta\hat{P}^\nu\end{aligned}\tag{D.22}$$

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